

ANALYTIC CONTINUATION OF HYPERGEOMETRIC FUNCTIONS IN THE RESONANT CASE

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ABSTRACT. We perform the analytic continuation of solutions to the hypergeometric differential equation of order n to the third regular singularity, usually denoted $z = 1$, with the help of recurrences of their Mellin–Barnes integral representations. In the resonant case, there are necessarily logarithmic solutions. We apply the result to Picard–Fuchs equations of certain one–parameter families of Calabi–Yau manifolds, known as the mirror quartic and the mirror quintic.

1. INTRODUCTION

Although monodromy and analytic continuation properties of the solutions of the hypergeometric differential equation have been well–studied in the past, interest got revived by mirror symmetry [3, 1]. In particular, Picard–Fuchs equations for one–parameter families of Calabi–Yau manifolds of dimension $n - 1$ whose Gauss–Manin connection has three regular singular points, one of which is of maximal unipotent monodromy, is a hypergeometric differential equation of order n with resonant exponents. In the case $n = 4$, such differential equations have been classified in [5].

The three regular singular points are usually taken to be $z = 0, 1, \infty$. The analytic continuation of the solutions from $z = 0$ to $z = \infty$ is well–known due to the fact that the differential equation after transformation to $w = \frac{1}{z}$ is again of hypergeometric form. The analytic continuation of the solutions from $z = 0$ to the $z = 1$ is more difficult to obtain since the differential equation after the transformation to $y = 1 - z$ is not of hypergeometric form for $n > 2$. Nevertheless, it is known in the nonresonant case, i.e. if there are no logarithmic solutions at 0 [14, 2]. We present here a complete solution to the problem of analytic continuation to $z = 1$ in the resonant case. Before we discuss the solution, let us present two applications of interest in the context of mirror symmetry. Let $\Phi_0(z), \Phi_1(z)$ denote certain chosen fundamental matrices of the hypergeometric differential equation near $z = 0, 1$. The variation of polarized Hodge structure of the family $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$ of mirror quartics given as

$$\mathcal{X}_z = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4z^{-\frac{1}{4}}x_0x_1x_2x_3 = 0\} \subset \mathbb{P}^3$$

leads to a hypergeometric differential equation of order 3. Then we prove

Theorem 7.1. *The analytic continuation of $\Phi_0(z)$ to $z = 1$ is determined by $\Phi_0(z) = \Phi_1(1 - z)M_{10}$ with*

$$M_{10} = \begin{pmatrix} \frac{A}{2\sqrt{2\pi}} & -\frac{A}{4\pi i} & 0 \\ \frac{2}{\sqrt{2\pi}}\left(\frac{3A}{64} + \frac{1}{A}\right) & -\frac{1}{\pi i}\left(\frac{3A}{64} - \frac{1}{A}\right) & 0 \\ -\frac{2}{\sqrt{2\pi}} & 0 & -\frac{1}{\sqrt{2\pi}} \end{pmatrix}$$

where $A = \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}$.

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Similarly, the variation of polarized Hodge structure of the family $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$ of mirror quintics given as

$$\mathcal{X}_z = \{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5z^{-\frac{1}{5}}x_0x_1x_2x_3x_4 = 0\} \subset \mathbb{P}^4$$

leads to a hypergeometric differential equation of order 4. Then, we prove

Theorem 7.2. *Let $\Phi_{z_0}(z)$ be the fundamental matrices near $z_0 = 0, 1$. The analytic continuation of $\Phi_0(z)$ to $z = 1$ is determined by $\Phi_0(z) = \Phi_1(1 - z)M_{10}$ with*

$$M_{10} = \begin{pmatrix} l_0 & -\frac{h_0}{2\pi i} & \frac{5k_0}{(2\pi i)^2} & 0 \\ w_0 & -\frac{h_1}{2\pi i} & \frac{5k_1}{(2\pi i)^2} & 2\pi i \\ 1 & 0 & 0 & 0 \\ w_1 - \frac{7}{10}w_0 & -\frac{h_2}{2\pi i} + \frac{7}{10}\frac{h_1}{2\pi i} & \frac{5k_2}{(2\pi i)^2} - \frac{7}{10}\frac{5k_1}{(2\pi i)^2} & 0 \end{pmatrix}$$

The real constants $l_0, w_0, w_1, h_0, h_1, h_2$ and the complex constants k_0, k_1, k_2 have an analytic expression which is not very simple, and therefore are given in the main text in (7.2), (7.3) and (7.4). This result might be useful to describe the embedding of the parameter space \mathbb{P}^1 into the period domain of the family \mathcal{X} .

We summarize how these results are obtained. In Section 2 we briefly review the known bases of solutions to the hypergeometric differential equations following [14] in terms of integral representations of the Mellin–Barnes type. We define the notion of resonant exponents. In the resonant case, we introduce a new basis of solutions $G_p(z)$ defined in terms of a Mellin–Barnes integral as

$$G_p(z) := \int \frac{dt}{2\pi i} e^{i\pi(p-2)t} z^t \prod_{j=1}^n \frac{\Gamma(\alpha_j + t)}{\Gamma(1 - \gamma_j + t)} \prod_{h=1}^p \Gamma(\gamma_h - t) \Gamma(1 - \gamma_h + t)$$

These solutions will be shown to have the following straightforward power series expansion at $z = 1$.

Theorem 6.5. *For any $2 < p \leq n$, if $|z - 1| < 1$, $\Re\beta_n > \Re\beta_p$, $\Re(\alpha_s + \gamma_j) > 0$, $j = 1, \dots, p$, $s = p + 1, \dots, n$, $\alpha_p + \gamma_p, \alpha_s + \gamma_{s+1} \notin \mathbb{Z}_{\leq 0}$, $s = 2, \dots, p - 1$ then*

$$\begin{aligned} G_p(z) &= \sum_{m=0}^{\infty} \Gamma(\alpha_1 + \gamma_2) \int \frac{dv}{2\pi i} e^{-i\pi v} \Gamma(\alpha_1 + \gamma_1 + v) \Gamma(-v) \\ &\quad \cdot \int \frac{ds}{2\pi i} \frac{B_{p,m}(s)}{\Gamma(m+1)} \frac{\Gamma(\gamma_2 - s) \Gamma(\gamma_1 + v - s)}{\Gamma(\alpha_1 + \gamma_1 + \gamma_2 + v - s)} \\ &\quad \cdot \int \frac{du}{2\pi i} e^{-i\pi u} \frac{\Gamma(-v + u) \Gamma(-u)}{\Gamma(-v)} \prod_{s=p+1}^n \frac{\Gamma(\alpha_s + \gamma_1 + u)}{\Gamma(1 - \gamma_s + \gamma_1 + u)} (1 - z)^m \end{aligned}$$

If $p = 2$ then

$$\begin{aligned} G_2(z) &= \sum_{m=0}^{\infty} \frac{\Gamma(\alpha_1 + \gamma_2 + m) \Gamma(\alpha_2 + \gamma_2 + m)}{\Gamma(m+1)} \\ &\quad \cdot \int \frac{dv}{2\pi i} e^{-i\pi v} \frac{\Gamma(\alpha_1 + \gamma_1 + v) \Gamma(\alpha_2 + \gamma_1 + v) \Gamma(-v)}{\Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + m + v)} \\ &\quad \cdot \int \frac{du}{2\pi i} e^{-i\pi u} \frac{\Gamma(-v + u) \Gamma(-u)}{\Gamma(-v)} \prod_{s=3}^n \frac{\Gamma(\alpha_s + \gamma_1 + u)}{\Gamma(1 - \gamma_s + \gamma_1 + u)} (1 - z)^m \end{aligned}$$

The functions $B_{p,m}(s)$ are given in terms of multiple Mellin–Barnes integrals.

The strategy to prove these statements is to find recurrences for all the solutions. The Mellin–Barnes integral representation of a solution to an order n equation is written in terms of a Mellin–Barnes integral representation of a solution to an

order $n - 1$ equation. In this way, the problem of analytic continuation is reduced to the one of solutions of an order 2 equation. The latter is well-known and briefly reviewed in Section 3. In Section 4 we collect the known recurrences due to [14] and [2]. We reformulate the latter in terms of integral representations and prove a new recurrence for the solutions $G_p(z)$. Equipped with these results, we perform the analytic continuation on the level of integral representations in Section 5. In order to make contact to the bases of solutions obtained from the Frobenius method in terms of power series expansions, we expand in Section 6 the integral representations obtained in Section 5 in powers of $1 - z$. Finally, in Section 7 we apply these series expansions to the examples presented above, the Picard–Fuchs equations of the family of mirror quartics and of the family of mirror quintics.

We wish to emphasize that in Sections 2 to 5 all results are formulated in terms of integral representations. At no point, such an integral is evaluated and converted into a power series. This will only be done in Section 6 in order to be able to discuss the explicit examples afterwards. For this reason, and also to make the presentation self-contained we include proofs of some of the results due to [14] and [2].

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2. THE HYPERGEOMETRIC EQUATION

2.1. The hypergeometric differential equation and its solutions. In this subsection we review the properties of the hypergeometric differential equation and its solutions that will be needed later on. For more background see e.g. [6, 16, 8]. We will, however, follow the notation of Nørlund [14].

We consider the hypergeometric differential equation of order n

$$(2.1) \quad \left(\theta \prod_{j=1}^{n-1} (\theta - \gamma_j) - z \prod_{j=1}^n (\theta - \alpha_j) \right) y(z) = 0$$

where $\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_{n-1} \in \mathbb{C}$ and $\theta = z \frac{d}{dz}$. For convenience, to make subsequent expressions more symmetric, we also introduce γ_n and set $\gamma_n = 0$. This differential equation has regular singularities at $z_0 = 0, 1, \infty$ with exponents

$$\begin{array}{l|cccccc} z_0 = 0 & \gamma_1 & \gamma_2 & \gamma_3 & \dots & \gamma_{n-1} & 0 \\ z_0 = 1 & 0 & 1 & 2 & \dots & n-2 & \beta_n \\ z_0 = \infty & \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{n-1} & \alpha_n \end{array}$$

where we have set

$$\beta_n = n - 1 - \sum_{i=1}^n (\alpha_i + \gamma_i).$$

Furthermore we define

$$f_n(t) = \prod_{j=1}^n \frac{\Gamma(\alpha_j + t)}{\Gamma(1 - \gamma_j + t)}$$

Definition 2.1. Let $q \in \{2, \dots, n\}$ and E be a maximal subset of exponents $\{\gamma_{i_1}, \dots, \gamma_{i_q}\}$ at $z_0 = 0$, or of exponents $\{\alpha_{i_1}, \dots, \alpha_{i_q}\}$ at $z_0 = \infty$, with the property that $\lambda - \mu \in \mathbb{Z}$ for all $\lambda, \mu \in E$. Then E is called to be in resonance or resonant. If there is no resonant subset of a set of exponents, we call the set of exponents nonresonant.

- Remark 2.2.** (1) If $\gamma_1, \dots, \gamma_q$ are resonant, Nørnlund uses the terminology that they “form a group”. Since they do not – mathematically speaking – form a group, we prefer the notion of resonance which is well known in the context of GKZ hypergeometric systems [7, 17]. The latter are a natural generalization of the hypergeometric differential equation discussed here.
- (2) The differential equation (2.1) remains invariant if we interchange α_j and γ_j , $j = 1, \dots, n$ and replace z by $\frac{1}{z}$. For simplicity, we will therefore restrict ourselves to resonant subsets of $\{\gamma_1, \dots, \gamma_n\}$.
- (3) In the following we only deal with the case that there is a single subset E of $\{\gamma_1, \dots, \gamma_n\}$ which is in resonance. We reorder the exponents such that $E = \{\gamma_1, \dots, \gamma_q\}$. If $\{\gamma_1, \dots, \gamma_n\}$ contains more than one resonant subset, the statements in this article are to be applied to each subset separately.
- (4) If $z = 0$ is a point of maximal unipotent monodromy, then $\gamma_i = 0$, $i = 1, \dots, n$, and hence $E = \{\gamma_1, \dots, \gamma_n\}$.

Next, we discuss bases of solutions to (2.1) near each singularity in terms of Mellin–Barnes integrals following Nørnlund.

- Near $z_0 = 0$:

If $\gamma_1, \dots, \gamma_n$ are not in resonance, Nørnlund gives the basis

$$(2.2) \quad \begin{aligned} y_j^*(z) &:= f_n(\gamma_j) z^{\gamma_j} {}_nF_{n-1} \left(\begin{matrix} \alpha_1 + \gamma_j, \dots, \alpha_n + \gamma_j \\ 1 - \gamma_1 + \gamma_j, \widehat{\dots}, 1 - \gamma_n + \gamma_j \end{matrix}; z \right) \\ &:= \int \frac{dt}{2\pi i} e^{-i\pi t} z^t f_n(t) \Gamma(\gamma_j - t) \Gamma(1 - \gamma_j + t), \quad j = 1, \dots, n \end{aligned}$$

for $|\arg(-z)| < \pi$. The $\widehat{}$ denotes the omission of the parameter $1 - \gamma_j + \gamma_j$.

If $\gamma_1, \dots, \gamma_q$, $2 \leq q \leq n$, are in resonance, then the solutions $y_j^*(z)$ are all equal for $1 \leq j \leq q$ and have to be replaced by

$$(2.3) \quad y_j^*(z) := \int dt z^t f_n(t) \frac{1}{(1 - e^{2\pi i(t - \gamma_1)})^j} \quad j = 1, \dots, q$$

for $0 < \arg(z) < 2\pi j$. Nørnlund shows that after this replacement for each resonant subset, the functions $y_j^*(z)$ form a linearly independent set of solutions. Note that $y_1^*(z)$ is the same for both cases. Furthermore, if we evaluate the integrals using the residue theorem, we obtain series expansion containing polynomials in $\log z$. Hence, we will refer to these solutions as the logarithmic solutions.

- Near $z_0 = \infty$:

As already mentioned, the differential equation (2.1) remains invariant if we interchange α_j and γ_j , $j = 1, \dots, n$ and replace z by $\frac{1}{z}$. A basis of solutions is given by

$$(2.4) \quad \begin{aligned} \bar{y}_j^*(z) &:= \prod_{k=1}^n \frac{\Gamma(\gamma_k + \alpha_j)}{\Gamma(1 - \alpha_k + \alpha_j)} z^{\alpha_j} {}_nF_{n-1} \left(\begin{matrix} \gamma_1 + \alpha_j, \dots, \gamma_n + \alpha_j \\ 1 - \alpha_1 + \alpha_j, \widehat{\dots}, 1 - \alpha_n + \alpha_j \end{matrix}; z \right) \\ &:= \int \frac{dt}{2\pi i} e^{-i\pi t} z^t \prod_{k=1}^n \frac{\Gamma(\gamma_k + t)}{\Gamma(1 - \alpha_k + t)} \Gamma(\alpha_j - t) \Gamma(1 - \alpha_j + t), \end{aligned}$$

for $j = 1, \dots, n$ and $|\arg(-z)| < \pi$.

- Near $z_0 = 1$:

If $\gamma_1, \dots, \gamma_n$ are not in resonance, a basis of solutions can be described as follows. There is the special solution corresponding to the exponent β_n . Nørnlund denotes this solution $\xi_n(z)$ if $\beta_n \notin \mathbb{Z}_{<0}$ and $\eta_n(z)$ otherwise. We

have

$$\xi_n(z) := \xi_n \left(\begin{matrix} \alpha_1, \dots, \alpha_n \\ \gamma_1, \dots, \gamma_n \end{matrix}; z \right) := z^{\gamma_1} (1-z)^{\beta_n} \sum_{k=0}^{\infty} c_k (1-z)^k$$

$$\eta_n(z) := z^{\gamma_1} \sum_{k=0}^{\infty} \frac{c_{k-\beta_n}}{\Gamma(k+1)} (1-z)^k$$

omitting the parameters when no confusion is possible. The coefficients c_k are determined recursively by the differential operator. Explicit formulas can be found in [14]. If in $\xi_n(z)$ we interchange α_j and γ_j , $j = 1, \dots, n$, and replace z by $\frac{1}{z}$, we obtain a solution which we denote by $\bar{\xi}_n(z)$ and differs from $\xi_n(z)$ by a factor $e^{\pm i\pi\beta_n}$, similarly for $\eta_n(z)$. Furthermore, there are $n-1$ holomorphic solutions which again according to Nørlund can be taken to be (for fixed i)

$$y_{ij}(z) = \frac{\pi}{\sin \pi(\gamma_j - \gamma_i)} (y_j^*(z) - y_i^*(z))$$

$$= \int \frac{dt}{2\pi i} z^t f_n(t) \frac{\pi}{\sin \pi(\gamma_j - t)} \frac{\pi}{\sin \pi(\gamma_i - t)},$$

for $j = 1, \dots, i-1, i+1, \dots, n$ and $y_j^*(z)$ as in (2.2). The integral is convergent for $-2\pi < \arg z < 2\pi$ and therefore represents a solution which is regular at $z = 1$. Nørlund only discusses these solutions when $\gamma_i - \gamma_j \notin \mathbb{Z}$. If $\gamma_i - \gamma_j \in \mathbb{Z}$ we apparently divide 0 by 0, but the quotient is well-defined as is easily seen from the integral representation. If there are three or more exponents in resonance, then the corresponding y_{ij} become linearly dependent. In the next subsection, we will give a basis of solutions in the resonant case in terms of integral representations, as well.

2.2. Meijer G-functions. We consider a special instance of the Meijer G-function [6, §5.3]

$$G_{n,n}^{p,n} \left(\begin{matrix} 1 - \alpha_1, \dots, 1 - \alpha_n \\ \gamma_1, \dots, \gamma_n \end{matrix}; z \right) = \int \frac{dt}{2\pi i} z^t \prod_{j=1}^n \frac{\Gamma(\alpha_j + t)}{\Gamma(1 - \gamma_j + t)} \prod_{h=1}^p \Gamma(\gamma_h - t) \Gamma(1 - \gamma_h + t)$$

for $1 \leq p \leq n$. This integral converges for $|\arg z| < p\pi$. We introduce the notation

$$(2.5) \quad G_p(z) := G_p \left(\begin{matrix} \alpha_1, \dots, \alpha_n \\ \gamma_1, \dots, \gamma_n \end{matrix}; z \right) := G_{n,n}^{p,n} \left(\begin{matrix} 1 - \alpha_1, \dots, 1 - \alpha_n \\ \gamma_1, \dots, \gamma_n \end{matrix}; (-1)^{p-2} z \right)$$

again omitting the parameters when no confusion is possible. These integrals define functions that are holomorphic (but not necessarily single-valued) at $z = 0$ for $p \geq 1$ and holomorphic, single-valued at $z = 1$ for $p > 1$. Moreover, they are also solutions to the hypergeometric differential equation (for $\gamma_n = 0$). In particular, if $\gamma_1, \dots, \gamma_n$ are not in resonance, and if we reorder the γ_i 's such that γ_j is in the first position, then

$$G_1 \left(\begin{matrix} \alpha_1, \dots, \alpha_n \\ \gamma_j, \gamma_1, \dots, \gamma_n \end{matrix}; z \right) = y_j^*(z)$$

with $y_j^*(z)$ as in (2.2) and if we reorder the γ_i 's such that γ_i, γ_j are in the first two positions we have,

$$G_2 \left(\begin{matrix} \alpha_1, \dots, \alpha_n \\ \gamma_i, \gamma_j, \gamma_1, \dots, \gamma_n \end{matrix}; z \right) = y_{ij}(z)$$

More generally, we have

Lemma 2.3. (1) If $\gamma_1, \dots, \gamma_n$ are not in resonance, then

$$G_p(z) = \sum_{j=1}^p \prod_{\substack{k=1 \\ k \neq j}}^p \frac{\pi e^{i\pi(p-1)\gamma_j}}{\sin \pi(\gamma_j - \gamma_k)} y_j^*(z)$$

with $y_j^*(z)$ as in (2.2).

(2) If $\gamma_1, \dots, \gamma_q$, $2 \leq q \leq n$, are resonant, then for $1 \leq p \leq q$:

$$G_p(z) = e^{-2\pi i \gamma_1} e^{i\pi \sum_{j=1}^p \gamma_j} (2\pi i)^{p-1} \sum_{j=1}^p (-1)^{p-j} \binom{p-1}{p-j} y_j^*(z)$$

with $y_j^*(z)$ as in (2.3).

Proof. (1) This is essentially Meijer's lemma. We present a proof that does not involve evaluating the residues. Instead we use the following trigonometric identity. For $t \notin \{\gamma_1, \dots, \gamma_n\}$ we have:

$$(2.6) \quad \prod_{j=1}^p \frac{\pi}{\sin \pi(\gamma_j - t)} = \sum_{j=1}^p \prod_{\substack{k=1 \\ k \neq j}}^p \frac{\pi}{\sin \pi(\gamma_j - \gamma_k)} \frac{\pi e^{(p-1)\pi i(\gamma_j - t)}}{\sin \pi(\gamma_j - t)}$$

This identity can be proven either by induction on p or by using a partial fraction expansion. Then, by consecutively applying Euler's identity and (2.6) to (2.5), we have

$$\begin{aligned} G_p(z) &= \int \frac{dt}{2\pi i} f_n(t) z^t e^{i\pi(p-2)t} \prod_{j=1}^p \frac{\pi}{\sin \pi(\gamma_j - t)} \\ &= \int \frac{dt}{2\pi i} f_n(t) z^t e^{i\pi(p-2)t} \sum_{j=1}^p \prod_{\substack{k=1 \\ k \neq j}}^p \frac{\pi}{\sin \pi(\gamma_j - \gamma_k)} \frac{\pi e^{i\pi(p-1)(\gamma_j - t)}}{\sin \pi(\gamma_j - t)} \\ &= \sum_{j=1}^p \prod_{\substack{k=1 \\ k \neq j}}^p \frac{\pi e^{i\pi(p-1)\gamma_j}}{\sin \pi(\gamma_j - \gamma_k)} \int \frac{dt}{2\pi i} f_n(t) z^t e^{-i\pi t} \frac{\pi}{\sin \pi(\gamma_j - t)} \end{aligned}$$

By (2.2), the last equation gives the claim.

(2) It is easy to see that

$$\begin{aligned} &\sum_{j=1}^p (-1)^{p-j} \binom{p-1}{p-j} \int dt z^t f_n(t) \frac{1}{(1 - e^{2\pi i(t-\gamma_1)})^j} \\ &= \int dt z^t f_n(t) \frac{e^{2(p-1)\pi i(t-\gamma_1)}}{(1 - e^{2\pi i(t-\gamma_1)})^p} \end{aligned}$$

Using Euler's identity for the Gamma function

$$\Gamma(1+x)\Gamma(-x) = \frac{\pi}{\sin \pi(-x)} = \frac{2\pi i e^{i\pi x}}{1 - e^{2\pi i x}}$$

the expression on the right hand side can be written as

$$\frac{1}{(2\pi i)^p} \int dt z^t f_n(t) e^{(p-2)\pi i(t-\gamma_1)} (\Gamma(1-\gamma_1+t)\Gamma(\gamma_1-t))^p$$

Finally, since the $\gamma_1, \dots, \gamma_q$ are resonant, we have for $1 \leq j \leq q$

$$\Gamma(1-\gamma_1+t)\Gamma(t-\gamma_1) = (-1)^{\gamma_1-\gamma_j} \Gamma(1-\gamma_j+t)\Gamma(\gamma_j-t)$$

so that above expression becomes

$$\frac{e^{2\pi i \gamma_1} e^{-i\pi \sum_{j=1}^p \gamma_j}}{(2\pi i)^{p-1}} G_p(z)$$

□

Remark 2.4. Lemma 2.3(1) is also valid if there are resonant exponents $\gamma_{i_1}, \dots, \gamma_{i_p}$. In this case, there is a zero in the numerator due to $y_{i_1}^*(z) = \dots = y_{i_p}^*(z)$ with $y_{i_j}^*(z)$ given in (2.2), and a zero of the same order in the denominator due to the sine. However, the quotient is well-defined as one can see by a de l'Hôpital argument, or more easily by the trigonometric identity (2.6). The left-hand side is always well-defined.

Proposition 2.5. *If $\gamma_1, \dots, \gamma_q$ are in resonance, then $G_p(z)$, $1 \leq p \leq q$, together with the $y_j^*(z)$, $j = q+1, \dots, n$ form a basis of solutions to the hypergeometric differential equation at $z = 0$.*

Proof. By Nørlund's result, we know that $\gamma_j^*(z)$, $j = 1, \dots, q$ form a linearly independent set of solutions. The statement then follows from Lemma 2.3 (b). Indeed, because $\binom{p-1}{j-1} = 0$ for $j > p$ and $\binom{p-1}{p-1} = 1$, the matrix (A_{pj}) , $1 \leq j, p \leq q$ with

$$A_{pj} = e^{-2\pi i \gamma_1} e^{i\pi \sum_{j=1}^p \gamma_j} (2\pi i)^{p-1} (-1)^{p-j} \binom{p-1}{p-j}$$

is upper triangular, with non vanishing entries on the diagonal, hence invertible. \square

Corollary 2.6. (1) *If $\gamma_1, \dots, \gamma_n$ are not in resonance, then $y_{ij}(z)$, for fixed i , $j = 1, \dots, i-1, i+1, \dots, n$ and $\xi_n(z)$ form a basis of solutions at $z = 1$.*
 (2) *If $\gamma_1, \dots, \gamma_q$ are resonant, then $\xi_n(z)$, $G_p(z)$, $p = 2, \dots, q$, and $y_{1j}(z)$, $j = q+1, \dots, n$ form a basis of solutions at $z = 1$.*

Therefore, we will refer to the solutions $G_p(z)$, $p > 1$, in the resonant case as logarithmic solutions, as well.

Lemma 2.3, Proposition 2.5 and Corollary 2.6 almost completely yield the analytic continuation from $z = 0$ to $z = 1$. We only lack the relation of $\xi_n(z)$ to the solutions $y_j^*(z)$ at $z = 0$. This will be reviewed in Section 5.1. However, in order to compare to the power series solutions obtained from the Frobenius method, we need to determine the series expansions of $y_{ij}(z)$ in the nonresonant case, and $G_p(z)$ in the resonant case at $z = 1$. This will be discussed in Section 6.

For completeness, we finish this section by reviewing the analytic continuation from $z_0 = 0$ to $z_0 = \infty$. The solutions $y_j^*(z)$ at $z_0 = 0$ and $\overline{y}_k^*(z)$ at $z_0 = \infty$ are related by $y_j^*(z) = \sum_{k=1}^n M_{0\infty, jk} \overline{y}_k^*(z)$. The entries of the matrix $M_{0\infty}$ can be determined by converting the Gamma functions in the integral representations of $y_j^*(z)$ and $\overline{y}_k^*(z)$ into sine functions using the Euler identity and then applying a partial fraction expansion. Nørlund gives explicit formulas for the coefficients $M_{0\infty, jk}$ in both the resonant and nonresonant cases [14].

3. ANALYTIC CONTINUATION FOR $n = 2$

In this section we review the well-known results for the analytic continuation of the classical hypergeometric function. We present them on one hand for completeness, and on the other hand because we will need them in the higher order situation. The case $n = 2$ is special because it is the only case where the differential equation at $z = 1$ is also in a hypergeometric form. Indeed, it is easy to see that under the change of variables $z \rightarrow 1 - z$ in (2.1), the indices γ_1 and β_2 are interchanged.

Lemma 3.1. (1) *For $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$ such that $\Re(\lambda_j + \mu_j) \notin \mathbb{Z}_{\leq 0}$ for $i, j = 1, 2$,*

$$\begin{aligned} & \int \frac{dt}{2\pi i} \Gamma(\lambda_1 + t) \Gamma(\lambda_2 + t) \Gamma(\mu_1 - t) \Gamma(\mu_2 - t) \\ &= \frac{\Gamma(\lambda_1 + \mu_1) \Gamma(\lambda_1 + \mu_2) \Gamma(\lambda_2 + \mu_1) \Gamma(\lambda_2 + \mu_2)}{\Gamma(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)} \end{aligned}$$

(2) For $\lambda_1, \lambda_2, \mu, \nu \in \mathbb{C}$ such that $\Re(\nu - \lambda_1 - \lambda_2 - \mu) > 0$,

$$\begin{aligned} & \int \frac{ds}{2\pi i} e^{\pm \pi i s} \frac{\Gamma(\lambda_1 + s)\Gamma(\lambda_2 + s)\Gamma(\mu - s)}{\Gamma(\nu + s)} \\ &= e^{\pi i \mu} \frac{\Gamma(\lambda_1 + \mu)\Gamma(\lambda_2 + \mu)\Gamma(\nu - \lambda_1 - \lambda_2 - \mu)}{\Gamma(\nu - \lambda_1)\Gamma(\nu - \lambda_2)}. \end{aligned}$$

Proof. The first identity is known as Barnes' first lemma. For a nice proof of both identities only involving integral representations see [11]. \square

Remark 3.2. As a consequence of Lemma 3.1(2) we obtain the well-known identity of Gauss:

$$(3.1) \quad {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix}; 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

valid for $\Re(c-a-b) > 0$.

Proposition 3.3.

$$\begin{aligned} & \int \frac{ds}{2\pi i} z^s e^{-i\pi s} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} \\ &= \frac{1}{\Gamma(c-a)\Gamma(c-b)} \int \frac{dt}{2\pi i} \Gamma(a+t)\Gamma(b+t)\Gamma(c-a-b-t)\Gamma(-t)(1-z)^t \end{aligned}$$

or in other words

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix}; z \right) = \frac{1}{\Gamma(c-a)\Gamma(c-b)} G_2 \left(\begin{matrix} a, b \\ c-a-b, 0 \end{matrix}; 1-z \right)$$

Proof. This is a well-known result. For the sake of exposition we give a proof. Setting $\lambda_1 = a, \lambda_2 = b, \mu_1 = s, \mu_2 = c-a-b$, Barnes' first lemma 3.1(1) can be rewritten as

$$\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} = \frac{1}{\Gamma(c-a)\Gamma(c-b)} \int_{k-i\infty}^{k+i\infty} \frac{dt}{2\pi i} \Gamma(a+t)\Gamma(b+t)\Gamma(s-t)\Gamma(c-a-b-t)$$

where we have chosen the integration contour as the line $\Re t = k$ for $0 < k < 1$, possibly indented such that poles of $\Gamma(a+t)$ and $\Gamma(b+t)$ lie to the right, and the poles of $\Gamma(s-t)$ and $\Gamma(c-a-b-t)$ lie to the left of this line. The left hand side of the claim then becomes

$$\begin{aligned} & \int \frac{ds}{2\pi i} z^s e^{-i\pi s} \frac{\Gamma(-s)}{\Gamma(c-a)\Gamma(c-b)} \\ & \cdot \int_{k-i\infty}^{k+i\infty} \frac{dt}{2\pi i} \Gamma(a+t)\Gamma(b+t)\Gamma(s-t)\Gamma(c-a-b-t) \end{aligned}$$

If k is chosen such that the lower bound of the distance between the s contour and the t contour is positive, i.e. nonzero, then the order of integration may be interchanged. Then the double integral takes the form

$$\begin{aligned} & \frac{1}{\Gamma(c-a)\Gamma(c-b)} \int_{k-i\infty}^{k+i\infty} \frac{dt}{2\pi i} \Gamma(a+t)\Gamma(b+t)\Gamma(c-a-b-t) \\ & \cdot \int \frac{ds}{2\pi i} z^s e^{-i\pi s} \Gamma(-s)\Gamma(s-t) \end{aligned}$$

Now the s integral can be evaluated using the identity [6, 16]

$$(3.2) \quad \int \frac{ds}{2\pi i} z^s e^{-i\pi s} \Gamma(-s)\Gamma(s-t) = \Gamma(-t)(1-z)^t \quad \square$$

From this proposition, the analytic continuation can be performed for all possible values of a, b, c , in particular for $c \in \mathbb{Z}$, and/or $c - a - b \in \mathbb{Z}$ which correspond to the resonant situation. We refer to [15] for details, where the function $\frac{(-1)^{c-1}\Gamma(c)}{\Gamma(a)\Gamma(b)}G_2\left(\begin{smallmatrix} a, b \\ 1-c, 0 \end{smallmatrix}; z\right)$ was denoted $g(a, b, c; z)$.

4. RECURRENCE RELATIONS BY ORDER

The general strategy for the analytic continuation of ... is the observation that each of the solutions $y_i^*(z), y_{ij}(z), G_p(z), \xi_n(z)$ given in Section 2 can be expressed in terms of the solutions of the same type of hypergeometric equations of order one less. This recurrence allows us to reduce the problem of the analytic continuation to the one of order two which is well-known, see Section 3. The reduction step will be discussed in Section 5.

4.1. Nørlund's recurrence for the solution ξ_n . We begin with the special solution at $z = 1$ associated to the exponent β_n . For $\beta_n \notin \mathbb{Z}_{<0}$ Nørlund has proven the following recurrence as well as several consequences [14]. These consequences will be used in Section 5.3.

Lemma 4.1. *If $\Re\beta_n > \Re\beta_{n-1} > -1$, then*

$$\xi_n(z) = \frac{\Gamma(\beta_n + 1)}{\Gamma(1 - \alpha_n - \gamma_n)\Gamma(\beta_{n-1} + 1)} z^{\gamma_n} \int_z^1 t^{\alpha_n-1} (t-z)^{-\alpha_n-\gamma_n} \xi_{n-1}(t) dt$$

Proof. A proof is given in [14]. It relies on showing that the generalized Euler integral on the right hand side is a solution to (2.1), and that its series expansion around $z = 1$ corresponds to the solution with index β_n . \square

Lemma 4.2. *If $\Re\beta_n > -1, \Re(x + \gamma_s) > 0, s = 1, \dots, n$, then*

$$\int_0^1 t^{x-1} \xi_n(t) dt = \frac{\Gamma(\beta_n + 1)\Gamma(x + \gamma_n)}{\Gamma(\beta_{n-1} + 1)\Gamma(x - \alpha_n + 1)} \int_0^1 u^{x-1} \xi_{n-1}(u) du$$

Proof. This is again due to [14]. The integral on the left hand side converges for $\Re\beta_n > -1, \Re(x + \gamma_s) > 0, s = 1, \dots, n$, since it is a linear function of the solutions $y_1^*(z), \dots, y_n^*(z)$. Substituting Lemma 4.1 yields

$$\begin{aligned} & \int_0^1 t^{x-1} \xi_n(t) dt \\ &= \frac{\Gamma(\beta_n + 1)}{\Gamma(1 - \alpha_n - \gamma_n)\Gamma(\beta_{n-1} + 1)} \int_0^1 t^{x-1+\gamma_n} \int_t^1 u^{\alpha_n-1} (u-t)^{-\alpha_n-\gamma_n} \xi_{n-1}(u) du dt \\ &= \frac{\Gamma(\beta_n + 1)}{\Gamma(1 - \alpha_n - \gamma_n)\Gamma(\beta_{n-1} + 1)} \int_0^1 u^{\alpha_n-1} \xi_{n-1}(u) \int_0^u t^{x-1+\gamma_n} (u-t)^{-\alpha_n-\gamma_n} dt du \\ &= \frac{\Gamma(\beta_n + 1)}{\Gamma(1 - \alpha_n - \gamma_n)\Gamma(\beta_{n-1} + 1)} \int_0^1 u^{x-1} \xi_{n-1}(u) du \int_0^1 t^{x-1+\gamma_n} (1-t)^{-\alpha_n-\gamma_n} dt \\ &= \frac{\Gamma(\beta_n + 1)}{\Gamma(1 - \alpha_n - \gamma_n)\Gamma(\beta_{n-1} + 1)} \frac{\Gamma(x + \gamma_n)\Gamma(-\alpha_n - \gamma_n + 1)}{\Gamma(x - \alpha_n + 1)} \int_0^1 u^{x-1} \xi_{n-1}(u) du \\ &= \frac{\Gamma(\beta_n + 1)\Gamma(x + \gamma_n)}{\Gamma(\beta_{n-1} + 1)\Gamma(x - \alpha_n + 1)} \int_0^1 u^{x-1} \xi_{n-1}(u) du \end{aligned}$$

For the second equality, we have used a generalization of a formula of Dichichlet's for changing the order of integration [10]. Moreover, [14] shows that the condition $\Re\beta_n > -1$ can be relaxed to $\beta_n \notin \mathbb{Z}_{<0}$. \square

Lemma 4.3. For $1 \leq p < n$, and if $\Re \beta_n > \Re \beta_p$, $\Re(x + \alpha_s) > 0$, $s = p+1, \dots, n$, then

$$\int_0^z t^{x-1} \bar{\xi}_{n-p} \left(\frac{\alpha_{p+1}, \dots, \alpha_n}{\gamma_{p+1}, \dots, \gamma_n}; \frac{z}{t} \right) dt = z^x \Gamma(\beta_n - \beta_p) \prod_{s=p+1}^n \frac{\Gamma(\alpha_s + x)}{\Gamma(1 - \gamma_s + x)}$$

Proof. This is also due to [14]. Applying Lemma 4.2 $n-1$ times yields

$$\int_0^1 t^{x-1} \xi_n(t) dt = \Gamma(\beta_n + 1) \prod_{s=1}^n \frac{\Gamma(x + \gamma_s)}{\Gamma(x - \alpha_s + 1)}$$

for $\Re(\beta_n) > -1$ and $\Re(x + \gamma_s) > 0$, $s = 1, \dots, n$. Recall from Section 2.1 that interchanging α_j with γ_j , $j = 1, \dots, n$ and replacing t with $1/t$ turns $\xi_n(z)$ into $\bar{\xi}_n(z)$. Applying these operations together with the replacement of x by $-x$ to the above integral yields

$$\int_1^\infty t^{x-1} \bar{\xi}_n(t) dt = \Gamma(\beta_n + 1) \prod_{s=1}^n \frac{\Gamma(\alpha_s - x)}{\Gamma(1 - \gamma_s - x)}$$

for $\Re \beta_n > -1$, $\Re(\alpha_s - x) > 0$, $s = 1, \dots, n$. Replacing t by $\frac{z}{t}$ and x by $-x$ and exhibiting the dependence of $\bar{\xi}_n(z)$ on the parameters α and γ yields

$$\int_0^z t^{x-1} \bar{\xi}_n \left(\frac{\alpha_1, \dots, \alpha_n}{\gamma_1, \dots, \gamma_n}; \frac{z}{t} \right) dt = z^x \Gamma(\beta_n + 1) \prod_{s=1}^n \frac{\Gamma(\alpha_s + x)}{\Gamma(1 - \gamma_s + x)}$$

Replacing n by $n-p$ for $p < n$ and restricting to the last $n-p$ parameters $\alpha_{p+1}, \dots, \alpha_n$, $\gamma_{p+1}, \dots, \gamma_n$, β_{n-p} can be written as $\beta_n - \beta_p - 1$, and the claim follows. \square

Lemma 4.4. For $1 \leq p < n$, $v \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and if $\Re \beta_n > \Re \beta_p$, $\Re(x + \alpha_s) > 0$, $s = p+1, \dots, n$, then

$$\begin{aligned} & \int_0^z dt t^{\gamma_1-1} (1-t)^v \bar{\xi}_{n-p} \left(\frac{\alpha_{p+1}, \dots, \alpha_n}{\gamma_{p+1}, \dots, \gamma_n}; \frac{z}{t} \right) \\ &= \Gamma(\beta_n - \beta_p) z^{\gamma_1} \int \frac{du}{2\pi i} e^{-i\pi u} z^u \frac{\Gamma(-v+u)\Gamma(-u)}{\Gamma(-v)} \prod_{s=p+1}^n \frac{\Gamma(\alpha_s + \gamma_1 + u)}{\Gamma(1 - \gamma_s + \gamma_1 + u)} \end{aligned}$$

Proof. We apply the binomial identity (3.2) to $(1-t)^v$ in the integrand on the left hand side and obtain

$$\int \frac{du}{2\pi i} e^{-i\pi u} \frac{\Gamma(-u)\Gamma(u-v)}{\Gamma(-v)} \int_0^z dt t^{\gamma_1+u-1} \bar{\xi}_{n-p} \left(\frac{\alpha_{p+1}, \dots, \alpha_n}{\gamma_{p+1}, \dots, \gamma_n}; \frac{z}{t} \right)$$

The claim follows from Lemma 4.3 and analytic continuation. \square

In order to state the following two propositions, we choose p of the parameters $\alpha_1, \dots, \alpha_n$ and p of the parameters $\gamma_1, \dots, \gamma_n$, say for simplicity $\alpha_1, \dots, \alpha_p$ and $\gamma_1, \dots, \gamma_p$. Our results do not depend on this choice, however, the complexity of explicit calculations can depend on it. Then we consider the hypergeometric differential equation of order p with exponents $\alpha_1, \dots, \alpha_p$ and $\gamma_1, \dots, \gamma_p$. For its solutions we introduce the notation

$$\tilde{y}_j^*(z) := \prod_{s=1}^p \frac{\Gamma(\alpha_s + \gamma_j)}{\Gamma(1 - \gamma_s + \gamma_j)} z^{\gamma_j} {}_pF_{p-1} \left(\frac{\alpha_1 + \gamma_j, \dots, \alpha_p + \gamma_j}{1 - \gamma_1 + \gamma_j, \dots, 1 - \gamma_p + \gamma_j}; z \right)$$

for $1 \leq j \leq p$, as well as the special solution

$$(4.1) \quad \tilde{G}_p(z) := \tilde{G}_p \left(\frac{\alpha_1, \dots, \alpha_p}{\gamma_1, \dots, \gamma_p}; z \right) := G_{p,p}^{p,p} \left(\frac{1 - \alpha_1, \dots, 1 - \alpha_p}{\gamma_1, \dots, \gamma_p}; (-1)^{p-2} z \right)$$

Proposition 4.5. *For $1 \leq p \leq n-1$, $1 \leq j \leq p$, and if $\Re \beta_n > \Re \beta_p$, $\Re(\gamma_j + \alpha_s) > 0$, $s = p+1, \dots, n$, we have*

$$y_j^*(z) = \frac{1}{\Gamma(\beta_n - \beta_p)} \int_0^z \frac{dt}{t} \tilde{y}_j^*(t) \bar{\xi}_{n-p} \left(\frac{\alpha_{p+1}, \dots, \alpha_n}{\gamma_{p+1}, \dots, \gamma_n}; \frac{z}{t} \right)$$

Proof. We use the Mellin–Barnes integral representation for $\tilde{y}_j^*(t)$ to write

$$\begin{aligned} & \int_0^z \frac{dt}{t} \tilde{y}_j^*(t) \bar{\xi}_{n-p} \left(\frac{\alpha_{p+1}, \dots, \alpha_n}{\gamma_{p+1}, \dots, \gamma_n}; \frac{z}{t} \right) \\ &= \int_0^z \frac{dt}{t} \int \frac{du}{2\pi i} e^{-i\pi u} t^u f_p(u) \Gamma(\gamma_j - u) \Gamma(1 - \gamma_j + u) \bar{\xi}_{n-p} \left(\frac{\alpha_{p+1}, \dots, \alpha_n}{\gamma_{p+1}, \dots, \gamma_n}; \frac{z}{t} \right) \end{aligned}$$

Due to the conditions $\Re(\alpha_s + \gamma_j) > 0$, $s = p+1, \dots, n$, we can interchange the integrals and apply Lemma 4.3. Then this expression becomes

$$\begin{aligned} & \int \frac{du}{2\pi i} e^{-i\pi u} f_p(u) \Gamma(\gamma_j - u) \Gamma(1 - \gamma_j + u) \int_0^z \frac{dt}{t} t^u \bar{\xi}_{n-p} \left(\frac{\alpha_{p+1}, \dots, \alpha_n}{\gamma_{p+1}, \dots, \gamma_n}; \frac{z}{t} \right) \\ &= \int \frac{du}{2\pi i} e^{-i\pi u} f_p(u) \Gamma(\gamma_j - u) \Gamma(1 - \gamma_j + u) z^u \Gamma(\beta_n - \beta_p) \prod_{s=p+1}^n \frac{\Gamma(\alpha_s + u)}{\Gamma(1 - \gamma_s + u)} \\ &= \Gamma(\beta_n - \beta_p) \int \frac{du}{2\pi i} e^{-i\pi u} z^u f_n(u) \Gamma(\gamma_j - u) \Gamma(1 - \gamma_j + u) \end{aligned}$$

The integral in the last expression is $y_j^*(z)$. \square

Proposition 4.5 has the following analog for $G_p(z)$.

Proposition 4.6. *For any $1 \leq p < n$, and if $\Re \beta_n > \Re \beta_p$, $\Re(\alpha_s + \gamma_j) > 0$, $j = 1, \dots, p$, $s = p+1, \dots, n$ then*

$$G_p \left(\frac{\alpha_1, \dots, \alpha_n}{\gamma_1, \dots, \gamma_n}; z \right) = \frac{1}{\Gamma(\beta_n - \beta_p)} \int_0^z \frac{dt}{t} \tilde{G}_p \left(\frac{\alpha_1, \dots, \alpha_p}{\gamma_1, \dots, \gamma_p}; t \right) \bar{\xi}_{n-p} \left(\frac{\alpha_{p+1}, \dots, \alpha_n}{\gamma_{p+1}, \dots, \gamma_n}; \frac{z}{t} \right)$$

Proof. By Lemma 2.3(1) and Remark 2.4 we have $G_p(z) = \sum_{j=1}^p A_{pj} y_j^*(z)$ with $A_{pj} = \prod_{\substack{k=1 \\ k \neq j}}^p \frac{\pi e^{i\pi(p-1)\gamma_j}}{\sin \pi(\gamma_j - \gamma_k)}$. We apply Proposition 4.5 and interchange the finite sum and the integral to obtain

$$G_p(z) = \frac{1}{\Gamma(\beta_n - \beta_p)} \int_0^z \frac{dt}{t} \sum_{j=1}^p A_{pj} \tilde{y}_j^*(t) \bar{\xi}_{n-p} \left(\frac{\alpha_{p+1}, \dots, \alpha_n}{\gamma_{p+1}, \dots, \gamma_n}; \frac{z}{t} \right)$$

From Lemma 2.3(1) for $n = p$ and Remark 2.4 we also have $\tilde{G}_p(z) = \sum_{j=1}^p A_{pj} \tilde{y}_j^*(z)$. The claim follows. \square

For $p = 2$ and $\gamma_1 - \gamma_2 \notin \mathbb{Z}$ these two propositions are due to Nørlund [14].

4.2. Bühring's recurrence for ${}_nF_{n-1}$. In this Section we review Bühring's recurrence for ${}_nF_{n-1}(z)$ from the point of view of integral representations [2]. It is then applicable to each of the $y_j^*(z)$ in the nonresonant case (2.2), as well as to $G_1(z)$ in the resonant case. Since this requires the parameters of ${}_nF_{n-1}(z)$ to take various sets of different values, we use a_1, \dots, a_n and b_1, \dots, b_{n-1} , as well as

$$(4.2) \quad c = \sum_{j=1}^{n-1} b_j - \sum_{j=1}^n a_j.$$

instead of $\alpha_1, \dots, \alpha_n$, $1 - \gamma_1, \dots, 1 - \gamma_{n-1}$, and β_n , respectively.

Lemma 4.7.

$$\begin{aligned}
& {}_nF_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) \\
&= \frac{\Gamma(b_{n-2})\Gamma(b_{n-1})}{\Gamma(a_n)\Gamma(b_{n-1}-a_n)\Gamma(b_{n-2}-a_n)} \\
&\cdot \int \frac{dt}{2\pi i} e^{\pm\pi it} \frac{\Gamma(-t)\Gamma(b_{n-1}-a_n+t)\Gamma(b_{n-2}-a_n+t)}{\Gamma(b_{n-1}+b_{n-2}-a_n+t)} \\
&\cdot {}_{n-1}F_{n-2} \left(\begin{matrix} a_1, \dots, a_{n-1} \\ b_1, \dots, b_{n-3}, b_{n-1}+b_{n-2}-a_n+t \end{matrix}; z \right)
\end{aligned}$$

Proof. Consider the Mellin Barnes integral representation of ${}_nF_{n-1}(z)$ given in (2.2)

$$\frac{\Gamma(a_1)\cdots\Gamma(a_n)}{\Gamma(b_1)\cdots\Gamma(b_{n-1})} {}_nF_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) = \int \frac{ds}{2\pi i} z^s e^{-i\pi s} \frac{\Gamma(a_1+s)\cdots\Gamma(a_n+s)\Gamma(-s)}{\Gamma(b_1+s)\cdots\Gamma(b_{n-1}+s)}$$

We apply Lemma 3.1(2) with $\lambda_1 = b_{n-1} - a_n, \lambda_2 = b_{n-2} - a_n, \mu = 0, \nu = b_{n-1} + b_{n-2} - a_n + t, s = t$ to obtain

$$\begin{aligned}
& \frac{\Gamma(a_1)\cdots\Gamma(a_n)}{\Gamma(b_1)\cdots\Gamma(b_{n-1})} {}_nF_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) \\
&= \frac{1}{\Gamma(b_{n-1}-a_n)\Gamma(b_{n-2}-a_n)} \int \frac{ds}{2\pi i} z^s e^{-i\pi s} \frac{\Gamma(a_1+s)\cdots\Gamma(a_{n-1}+s)\Gamma(-s)}{\Gamma(b_1+s)\cdots\Gamma(b_{n-3}+s)} \\
&\cdot \int \frac{dt}{2\pi i} e^{\pm\pi it} \frac{\Gamma(b_{n-1}-a_n+t)\Gamma(b_{n-2}-a_n+t)\Gamma(-t)}{\Gamma(b_{n-1}+b_{n-2}-a_n+s+t)}
\end{aligned}$$

which is valid for $\Re(a_n+t) > 0$. This justifies interchanging the order of integration as in the proof of Proposition 3.3 and yields, after extension to the whole t plane by analytic continuation of the parameters, the desired result.

$$\begin{aligned}
& \frac{\Gamma(a_1)\cdots\Gamma(a_n)}{\Gamma(b_1)\cdots\Gamma(b_{n-1})} {}_nF_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) \\
&= \frac{1}{\Gamma(b_{n-1}-a_n)\Gamma(b_{n-2}-a_n)} \int \frac{dt}{2\pi i} e^{\pm\pi it} \Gamma(b_{n-1}-a_n+t)\Gamma(b_{n-2}-a_n+t)\Gamma(-t) \\
&\int \frac{ds}{2\pi i} z^s e^{-i\pi s} \frac{\Gamma(a_1+s)\cdots\Gamma(a_{n-1}+s)\Gamma(-s)}{\Gamma(b_1+s)\cdots\Gamma(b_{n-3}+s)\Gamma(b_{n-1}+b_{n-2}-a_n+s+t)} \\
&= \frac{\Gamma(a_1)\cdots\Gamma(a_{n-1})}{\Gamma(b_1)\cdots\Gamma(b_{n-3})\Gamma(b_{n-1}-a_n)\Gamma(b_{n-2}-a_n)} \\
&\cdot \int \frac{dt}{2\pi i} e^{\pm\pi it} \frac{\Gamma(-t)\Gamma(b_{n-1}-a_n+t)\Gamma(b_{n-2}-a_n+t)}{\Gamma(b_{n-1}+b_{n-2}-a_n+t)} \\
&\cdot {}_{n-1}F_{n-2} \left(\begin{matrix} a_1, \dots, a_{n-1} \\ b_1, \dots, b_{n-3}, b_{n-1}+b_{n-2}-a_n+t \end{matrix}; z \right)
\end{aligned}$$

□

Using this lemma, we find the following integral representation of ${}_nF_{n-1}(z)$ in terms of ${}_2F_1(z)$:

Proposition 4.8.

$$\begin{aligned}
& {}_nF_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) \\
&= \frac{\Gamma(b_{n-1})\cdots\Gamma(b_1)}{\Gamma(a_n)\cdots\Gamma(a_1)} \int \frac{du}{2\pi i} \tilde{A}^{(n)}(u) \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c+a_1+a_2+u)} {}_2F_1 \left(\begin{matrix} a_1, a_2 \\ c+a_1+a_2+u \end{matrix}; z \right)
\end{aligned}$$

with

$$\tilde{A}^{(n)}(u)$$

$$\begin{aligned}
&= \frac{1}{\Gamma(b_{n-1} - a_n)\Gamma(b_{n-2} - a_n)\Gamma(b_{n-3} - a_{n-1}) \cdots \Gamma(b_1 - a_3)} \\
&\cdot \int \frac{du_{n-2}}{2\pi i} e^{\pm \pi i u_{n-2}} \frac{\Gamma(-u_{n-2})\Gamma(b_{n-1} - a_n + u_{n-2})\Gamma(b_{n-2} - a_n + u_{n-2})}{\Gamma(b_{n-1} + b_{n-2} - a_n - a_{n-1} + u_{n-2})} \\
&\cdot \int \frac{du_{n-3}}{2\pi i} e^{\pm \pi i(u_{n-3} - u_{n-2})} \frac{\Gamma(u_{n-2} - u_{n-3})\Gamma(b_{n-1} + b_{n-2} - a_n - a_{n-1} + u_{n-3})}{\Gamma(b_{n-1} + b_{n-2} + b_{n-3} - a_n - a_{n-1} - a_{n-2} + u_{n-3})} \\
&\quad \cdot \Gamma(b_{n-3} - a_{n-1} + u_{n-3} - u_{n-2}) \\
&\quad \cdot \dots \\
&\cdot \int \frac{du_2}{2\pi i} e^{\pm \pi i(u_2 - u_3)} \frac{\Gamma(u_3 - u_2)\Gamma(b_{n-1} + \cdots + b_3 - a_n - \cdots - a_4 + u_2)}{\Gamma(b_{n-1} + \cdots + b_2 - a_n - \cdots - a_3 + u_2)} \\
&\quad \cdot \Gamma(b_2 - a_4 + u_2 - u_3) \\
&\cdot e^{\pm \pi i(u - u_2)} \Gamma(u_2 - u) \Gamma(c + a_1 + a_2 - b_1 + u) \Gamma(b_1 - a_3 + u - u_2)
\end{aligned}$$

For the applications in Section 7 we give the first few cases explicitly:

$$(4.3) \quad \tilde{A}^{(3)}(u) = e^{\pm \pi i u} \frac{\Gamma(c + a_1 + a_2 - b_1 + u) \Gamma(b_1 - a_3 + u) \Gamma(-u)}{\Gamma(b_2 - a_3) \Gamma(b_1 - a_3)}$$

$$\begin{aligned}
(4.4) \quad \tilde{A}^{(4)}(u) &= \frac{1}{\Gamma(b_3 - a_4) \Gamma(b_2 - a_4) \Gamma(b_1 - a_3)} \\
&\cdot \int \frac{du_2}{2\pi i} e^{\pm \pi i u_2} \frac{\Gamma(-u_2) \Gamma(b_3 - a_4 + u_2) \Gamma(b_2 - a_4 + u_2)}{\Gamma(b_3 + b_2 - a_4 - a_3 + u_2)} \\
&\cdot e^{\pm \pi i(u - u_2)} \Gamma(u_2 - u) \Gamma(c + a_1 + a_2 - b_1 + u) \Gamma(b_1 - a_3 + u - u_2)
\end{aligned}$$

Proof. Repeated application of Lemma 4.7 yields

$$\begin{aligned}
&{}_n F_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) \\
&= \frac{\Gamma(b_{n-1}) \Gamma(b_{n-2}) \Gamma(b_{n-3})}{\Gamma(a_n) \Gamma(a_{n-1}) \Gamma(b_{n-1} - a_n) \Gamma(b_{n-2} - a_n) \Gamma(b_{n-3} - a_{n-1})} \\
&\cdot \int \frac{dt_{n-2}}{2\pi i} e^{\pm \pi i t_{n-2}} \frac{\Gamma(-t_{n-2}) \Gamma(b_{n-1} - a_n + t_{n-2}) \Gamma(b_{n-2} - a_n + t_{n-2})}{\Gamma(b_{n-1} + b_{n-2} - a_n - a_{n-1} + t_{n-2})} \\
&\cdot \int \frac{dt_{n-3}}{2\pi i} e^{\pm \pi i t_{n-3}} \frac{\Gamma(-t_{n-3}) \Gamma(b_{n-1} + b_{n-2} - a_n + t_{n-2} - a_{n-1} + t_{n-3})}{\Gamma(b_{n-1} + b_{n-2} - a_n + t_{n-2} + b_{n-3} - a_{n-1} + t_{n-3})} \\
&\quad \cdot \Gamma(b_{n-3} - a_{n-1} + t_{n-3}) \\
&\quad \cdot {}_{n-2} F_{n-3} \left(\begin{matrix} a_1, \dots, a_{n-2} \\ b_1, \dots, b_{n-4}, b_{n-1} + b_{n-2} + b_{n-3} - a_n - a_{n-1} + t_{n-2} + t_{n-3} \end{matrix}; z \right) \\
&= \frac{\Gamma(b_{n-1}) \cdots \Gamma(b_1)}{\Gamma(a_n) \cdots \Gamma(a_3) \Gamma(b_{n-1} - a_n) \Gamma(b_{n-2} - a_n) \Gamma(b_{n-3} - a_{n-1}) \cdots \Gamma(b_1 - a_3)} \\
&\cdot \int \frac{dt_{n-2}}{2\pi i} e^{\pm \pi i t_{n-2}} \frac{\Gamma(-t_{n-2}) \Gamma(b_{n-1} - a_n + t_{n-2}) \Gamma(b_{n-2} - a_n + t_{n-2})}{\Gamma(b_{n-1} + b_{n-2} - a_n - a_{n-1} + t_{n-2})} \\
&\cdot \int \frac{dt_{n-3}}{2\pi i} e^{\pm \pi i t_{n-3}} \frac{\Gamma(-t_{n-3}) \Gamma(b_{n-1} + b_{n-2} - a_n - a_{n-1} + t_{n-2} + t_{n-3})}{\Gamma(b_{n-1} + b_{n-2} + b_{n-3} - a_n - a_{n-1} + t_{n-2} + t_{n-3})} \\
&\quad \cdot \Gamma(b_{n-3} - a_{n-1} + t_{n-3}) \\
&\quad \cdot \dots
\end{aligned}$$

$$\begin{aligned}
& \cdot \int \frac{dt_1}{2\pi i} e^{\pm \pi i t_1} \frac{\Gamma(-t_1) \Gamma(b_{n-1} + \cdots + b_2 - a_n - \cdots - a_3 + t_{n-2} + \cdots + t_1)}{\Gamma(b_{n-1} + \cdots + b_1 - a_n - \cdots - a_3 + t_{n-2} + \cdots + t_1)} \\
& \cdot \Gamma(b_1 - a_3 + t_1) \\
& \cdot {}_2F_1 \left(\begin{matrix} a_1, a_2 \\ b_{n-1} + \cdots + b_1 - a_n - \cdots - a_3 + t_{n-2} + \cdots + t_1 \end{matrix} ; z \right)
\end{aligned}$$

Introducing new integration variables by

$$u_k = \sum_{j=k}^{n-2} t_j, \quad k = 1, \dots, n-2$$

setting $u = u_1$ and using (4.2) yields the claim. \square

4.3. A recurrence for \tilde{G}_p . In this section we prove a new recurrence for the functions $\tilde{G}_p(z)$ that were defined in (4.1) and appeared in Proposition 4.6. The recurrence is similar to the recurrence for ${}_nF_{n-1}(z)$ in the last section.

Lemma 4.9. *For any $1 \leq p < n$ and if $\alpha_{p-1} + \gamma_p, \alpha_p + \gamma_p \notin \mathbb{Z}_{\leq 0}$, then*

$$\begin{aligned}
\tilde{G}_p \left(\begin{matrix} \alpha_1, \dots, \alpha_p \\ \gamma_1, \dots, \gamma_p \end{matrix} ; z \right) &= \Gamma(\alpha_{p-1} + \gamma_p) \Gamma(\alpha_p + \gamma_p) \\
& \cdot \int \frac{ds}{2\pi i} e^{\pm \pi i s} \frac{\Gamma(\alpha_{p-1} + s) \Gamma(\alpha_p + s)}{\Gamma(\alpha_{p-1} + \alpha_p + \gamma_p + s)} \tilde{G}_{p-1} \left(\begin{matrix} \alpha_1, \dots, \alpha_{p-2}, -s \\ \gamma_1, \dots, \gamma_{p-1} \end{matrix} ; z \right)
\end{aligned}$$

Proof. We apply Lemma 3.1(2) with $\lambda_1 = \alpha_{p-1}, \lambda_2 = \alpha_p, \mu = t, \nu = \alpha_{p-1} + \alpha_p + \gamma_p$ to obtain

$$\begin{aligned}
\tilde{G}_p \left(\begin{matrix} \alpha_1, \dots, \alpha_p \\ \gamma_1, \dots, \gamma_p \end{matrix} ; z \right) &= \Gamma(\alpha_{p-1} + \gamma_p) \Gamma(\alpha_p + \gamma_p) \int \frac{dt}{2\pi i} z^t \prod_{h=1}^{p-2} \Gamma(\alpha_h + t) \prod_{h=1}^{p-1} \Gamma(\gamma_h - t) \\
& \cdot \int \frac{ds}{2\pi i} e^{\pm \pi i s} \frac{\Gamma(t-s) \Gamma(\alpha_{p-1} + s) \Gamma(\alpha_p + s)}{\Gamma(\alpha_{p-1} + \alpha_p + \gamma_p + s)}
\end{aligned}$$

which is valid for $\Re(\gamma_p - t) > 0$. This justifies interchanging the order of integration as in the proof of Proposition 3.3 and yields, after extension to the whole t plane by analytic continuation of the parameters, the desired result.

$$\begin{aligned}
\tilde{G}_p \left(\begin{matrix} \alpha_1, \dots, \alpha_p \\ \gamma_1, \dots, \gamma_p \end{matrix} ; z \right) &= \Gamma(\alpha_{p-1} + \gamma_p) \Gamma(\alpha_p + \gamma_p) \int \frac{ds}{2\pi i} e^{\pm \pi i s} \frac{\Gamma(\alpha_{p-1} + s) \Gamma(\alpha_p + s)}{\Gamma(\alpha_{p-1} + \alpha_p + \gamma_p + s)} \\
& \cdot \int \frac{dt}{2\pi i} z^t \prod_{h=1}^{p-2} \Gamma(\alpha_h + t) \Gamma(-s + t) \prod_{h=1}^{p-1} \Gamma(\gamma_h - t)
\end{aligned}$$

\square

Proposition 4.10. *For $3 \leq p < n$, and if $\alpha_p + \gamma_p, \alpha_s + \gamma_{s+1} \notin \mathbb{Z}_{\leq 0}$, $s = 2, \dots, p-1$, then*

$$\tilde{G}_p \left(\begin{matrix} \alpha_1, \dots, \alpha_p \\ \gamma_1, \dots, \gamma_p \end{matrix} ; z \right) = \int \frac{ds}{2\pi i} B_p(s) \tilde{G}_2 \left(\begin{matrix} \alpha_1, -s \\ \gamma_1, \gamma_2 \end{matrix} ; z \right)$$

where

$$\begin{aligned}
B_p(s) &= \Gamma(\alpha_p + \gamma_p) \Gamma(\alpha_{p-1} + \gamma_p) \Gamma(\alpha_{p-2} + \gamma_{p-1}) \dots \Gamma(\alpha_2 + \gamma_3) \\
&\cdot \int \frac{ds_{p-2}}{2\pi i} e^{-i\pi s_{p-2}} \frac{\Gamma(\alpha_p + s_{p-2}) \Gamma(\alpha_{p-1} + s_{p-2}) \Gamma(\gamma_{p-1} + s_{p-2})}{\Gamma(\alpha_p + \alpha_{p-1} + \gamma_p + s_{p-2})} \\
&\cdot \int \frac{ds_{p-3}}{2\pi i} e^{-i\pi s_{p-3}} \frac{\Gamma(\alpha_{p-2} + s_{p-3}) \Gamma(\gamma_{p-2} + s_{p-3}) \Gamma(-s_{p-2} + s_{p-3})}{\Gamma(\alpha_{p-2} + \gamma_{p-1} - s_{p-2} + s_{p-3})} \\
&\cdot \dots \\
&\cdot \int \frac{ds_2}{2\pi i} e^{-i\pi s_2} \frac{\Gamma(\alpha_3 + s_2) \Gamma(\gamma_3 + s_2) \Gamma(-s_3 + s_2)}{\Gamma(\alpha_3 + \gamma_4 - s_3 + s_2)} \\
&\cdot e^{-i\pi s} \frac{\Gamma(\alpha_2 + s) \Gamma(-s_2 + s)}{\Gamma(\alpha_2 + \gamma_3 - s_2 + s)}
\end{aligned}$$

Proof. Repeated application of Lemma 4.9 immediately yields the result. \square

For the applications in Section 7 we give the first case explicitly.

$$(4.5) \quad B_3(s) = \Gamma(\alpha_3 + \gamma_3) \Gamma(\alpha_2 + \gamma_3) e^{-i\pi s} \frac{\Gamma(\alpha_2 + s) \Gamma(\alpha_3 + s)}{\Gamma(\alpha_2 + \alpha_3 + \gamma_3 + s)}$$

5. ANALYTIC CONTINUATION TO $z = 1$ FOR $n > 2$

In this section we discuss the analytic continuation from $z = 0$ to $z = 1$ for the solutions to the hypergeometric differential equation of arbitrary order $n > 2$. We first review the known results for the special solution $\bar{\xi}_n(z)$ and the holomorphic solutions ${}_nF_{n-1}(z)$, then we prove the theorem for the logarithmic solutions $G_p(z)$.

5.1. The solution ξ_n . For completeness and application in Section 7 we reproduce here Nørlunds beautiful formula for the analytic continuation of the special solution ξ_n .

Proposition 5.1. (1) *If $\gamma_1, \dots, \gamma_n$ are not in resonance, then*

$$\xi_n(z) = \Gamma(\beta_n + 1) \sum_{j=1}^n \frac{\prod_{k=1, k \neq j}^n \Gamma(\gamma_k - \gamma_j)}{\prod_{k=1}^n \Gamma(1 - \alpha_k - \gamma_j)} y_j^*(z)$$

(2) *If $\gamma_1, \dots, \gamma_q$ are in resonance, then*

$$\begin{aligned}
\xi_n(z) &= \frac{\Gamma(\beta_n + 1)}{2\pi i} \sum_{j=1}^q \frac{(-1)^j}{(q-j)!} \psi^{(q-j)}(e^{2\pi i \gamma_1}) e^{-2\pi i j \gamma_1} y_j^*(z) \\
&+ \frac{1}{\pi} \sum_{j=q+1}^n \frac{\prod_{k=1, k \neq j}^n \Gamma(\gamma_k - \gamma_j)}{\prod_{k=1}^n \Gamma(1 - \alpha_k - \gamma_j)} y_j^*(z)
\end{aligned}$$

where

$$\psi(x) = e^{-i\pi \beta_n} \frac{\prod_{k=1}^n (x - e^{-2\pi i \alpha_k})}{\prod_{k=q+1}^n (x - e^{2\pi i \gamma_k})}$$

Proof. We are not aware of a simple proof involving only integral representations. Therefore, we refer to the proof of [14] based on partial fraction decompositions and the special properties of the functions $\xi_n(z)$. \square

5.2. Holomorphic solutions. For completeness and application in Section 6 we reproduce here part of the result [2] for the analytic continuation of the holomorphic solutions ${}_nF_{n-1}(z)$ in terms of integral representations. The complete result will be given in Section 6.1 when we evaluate the following integral. In order to state the result, recall the value of c from (4.2).

Proposition 5.2.

$$\begin{aligned} & \frac{\Gamma(a_1) \cdots \Gamma(a_n)}{\Gamma(b_1) \cdots \Gamma(b_{n-1})} {}_nF_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) \\ &= \int \frac{dt}{2\pi i} \int \frac{du}{2\pi i} \tilde{A}^{(n)}(t) \frac{\Gamma(a_1+u)\Gamma(a_2+u)\Gamma(c+t-u)\Gamma(-u)}{\Gamma(c+a_2+t)\Gamma(c+a_1+t)} (1-z)^u \end{aligned}$$

Proof. This follows immediately from Propositions 4.8 and 3.3. \square

5.3. Logarithmic solutions. In this section we start to prove the main theorem. We give an integral representation for the solutions $G_p(z)$, $p > 1$ which admits an easy power series expansion about $z = 1$. This expansion will be discussed in Section 6.1. If $\{\gamma_1, \dots, \gamma_q\}$, $q > 1$, is resonant, then, as discussed in Section 2.1, have logarithms near $z = 0$.

Proposition 5.3. *For $2 < p \leq q \leq n$, and if $\Re \beta_n > \Re \beta_p$, $\Re(\alpha_s + \gamma_j) > 0$, $j = 1, \dots, p$, $s = p+1, \dots, n$, $\alpha_p + \gamma_p, \alpha_s + \gamma_{s+1} \notin \mathbb{Z}_{\leq 0}$, $s = 2, \dots, p-1$ then*

$$\begin{aligned} G_p(z) &= \Gamma(\alpha_1 + \gamma_2) \int \frac{dv}{2\pi i} e^{-i\pi v} \Gamma(\alpha_1 + \gamma_1 + v) \Gamma(-v) \\ &\quad \cdot \int \frac{ds}{2\pi i} B_p(s) \frac{\Gamma(\gamma_2 - s) \Gamma(\gamma_1 + v - s)}{\Gamma(\alpha_1 + \gamma_1 + \gamma_2 + v - s)} \\ &\quad \cdot z^{\gamma_1} \int \frac{du}{2\pi i} e^{-i\pi u} z^u \frac{\Gamma(-v+u) \Gamma(-u)}{\Gamma(-v)} \prod_{s=p+1}^n \frac{\Gamma(\alpha_s + \gamma_1 + u)}{\Gamma(1 - \gamma_s + \gamma_1 + u)} \end{aligned}$$

If $p = 2$, then

$$\begin{aligned} G_2(z) &= \Gamma(\alpha_1 + \gamma_2) \Gamma(\alpha_2 + \gamma_2) \int \frac{dv}{2\pi i} e^{-i\pi v} \frac{\Gamma(\alpha_1 + \gamma_1 + v) \Gamma(\alpha_2 + \gamma_1 + v) \Gamma(-v)}{\Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + v)} \\ &\quad \cdot z^{\gamma_1} \int \frac{du}{2\pi i} e^{-i\pi u} z^u \frac{\Gamma(-v+u) \Gamma(-u)}{\Gamma(-v)} \prod_{s=3}^n \frac{\Gamma(\alpha_s + \gamma_1 + u)}{\Gamma(1 - \gamma_s + \gamma_1 + u)} \end{aligned}$$

Proof. We first consider the case $p > 2$. We abbreviate the left-hand side by $G_p(z)$. By Proposition 4.6 we can have

$$G_p(z) = \frac{1}{\Gamma(\beta_n - \beta_p)} \int_0^z \frac{dt}{t} \tilde{G}_p \left(\begin{matrix} \alpha_1, \dots, \alpha_p \\ \gamma_1, \dots, \gamma_p \end{matrix}; t \right) \bar{\xi}_{n-p} \left(\begin{matrix} \alpha_{p+1}, \dots, \alpha_n \\ \gamma_{p+1}, \dots, \gamma_n \end{matrix}; \frac{z}{t} \right)$$

By Proposition 4.10 we can write this as

$$G_p(z) = \frac{1}{\Gamma(\beta_n - \beta_p)} \int_0^z \frac{dt}{t} \bar{\xi}_{n-p} \left(\begin{matrix} \alpha_{p+1}, \dots, \alpha_n \\ \gamma_{p+1}, \dots, \gamma_n \end{matrix}; \frac{z}{t} \right) \int \frac{ds}{2\pi i} B_p(s) G_2 \left(\begin{matrix} \alpha_1, -s \\ \gamma_1, \gamma_2 \end{matrix}; t \right)$$

Shifting the integration variable in G_2 yields

$$G_2 \left(\begin{matrix} \alpha_1, -s \\ \gamma_1, \gamma_2 \end{matrix}; t \right) = t^{\gamma_1} G_2 \left(\begin{matrix} \alpha_1 + \gamma_1, \gamma_1 - s \\ \gamma_2 - \gamma_1, 0 \end{matrix}; t \right)$$

By Proposition 3.3 with $a = \alpha_1 + \gamma_1, b = \gamma_1 - s, c = \alpha_1 + \gamma_1 + \gamma_2 - s$ and using the integral representation of ${}_2F_1$ we obtain

$$\begin{aligned} G_p(z) &= \frac{\Gamma(\alpha_1 + \gamma_2)}{\Gamma(\beta_n - \beta_p)} \int_0^z \frac{dt}{t} t^{\gamma_1} \bar{\xi}_{n-p} \left(\begin{matrix} \alpha_{p+1}, \dots, \alpha_n \\ \gamma_{p+1}, \dots, \gamma_n \end{matrix}; \frac{z}{t} \right) \int \frac{ds}{2\pi i} B_p(s) \Gamma(\gamma_2 - s) \\ &\quad \cdot \int \frac{dv}{2\pi i} (1-t)^v e^{-i\pi v} \frac{\Gamma(\alpha_1 + \gamma_1 + v) \Gamma(\gamma_1 - s + v) \Gamma(-v)}{\Gamma(\alpha_1 + \gamma_1 + \gamma_2 - s + v)} \end{aligned}$$

Under the conditions on $\alpha_i, \beta_j, \gamma_k$ stated in the claim, we can change the order of the integrals

$$\begin{aligned} G_p(z) &= \frac{\Gamma(\alpha_1 + \gamma_2)}{\Gamma(\beta_n - \beta_p)} \int \frac{dv}{2\pi i} e^{-i\pi v} \Gamma(\alpha_1 + \gamma_1 + v) \Gamma(-v) \\ &\quad \cdot \int \frac{ds}{2\pi i} B_p(s) \frac{\Gamma(\gamma_2 - s) \Gamma(\gamma_1 + v - s)}{\Gamma(\alpha_1 + \gamma_1 + \gamma_2 + v - s)} \\ &\quad \cdot \int_0^z \frac{dt}{t} t^{\gamma_1} (1-t)^v \bar{\xi}_{n-p} \left(\begin{matrix} \alpha_{p+1}, \dots, \alpha_n \\ \gamma_{p+1}, \dots, \gamma_n \end{matrix}; \frac{z}{t} \right) \end{aligned}$$

Finally, the t integral can be evaluated by Lemma 4.4. This yields the claim.

Now consider the case $p = 2$. The first step using Proposition 4.6 for general p applies here, too. However, for the second step we do not need the recursion formula from Proposition 4.10. The remaining steps are as above. We shift again the integration variable in G_2 and use Proposition 3.3 to get

$$\begin{aligned} G_2(z) &= \frac{\Gamma(\alpha_1 + \gamma_2) \Gamma(\alpha_2 + \gamma_2)}{\Gamma(\beta_n - \beta_2)} \int_0^z \frac{dt}{t} t^{\gamma_1} \bar{\xi}_{n-2} \left(\begin{matrix} \alpha_3, \dots, \alpha_n \\ \gamma_3, \dots, \gamma_n \end{matrix}; \frac{z}{t} \right) \\ &\quad \cdot \int \frac{dv}{2\pi i} (1-t)^v e^{-i\pi v} \frac{\Gamma(\alpha_1 + \gamma_1 + v) \Gamma(\alpha_2 + \gamma_1 + v) \Gamma(-v)}{\Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + v)} \end{aligned}$$

After changing the order of integration and evaluating the t integral with the help of Lemma 4.4 we obtain the claim. \square

Remark 5.4. By (2.2) the u integral in Theorem 5.3 is nothing but the integral representation of a hypergeometric function:

$$\begin{aligned} &z^{\gamma_1} \int \frac{du}{2\pi i} e^{-i\pi u} z^u \frac{\Gamma(-v+u) \Gamma(-u)}{\Gamma(-v)} \prod_{s=p+1}^n \frac{\Gamma(\alpha_s + \gamma_1 + u)}{\Gamma(1 - \gamma_s + \gamma_1 + u)} \\ &= z^{\gamma_1} \prod_{k=p+1}^n \frac{\Gamma(\alpha_k + \gamma_1)}{\Gamma(1 - \gamma_k + \gamma_1)} {}_{n-p+1}F_{n-p} \left(\begin{matrix} -v, \alpha_{p+1} + \gamma_1, \dots, \alpha_n + \gamma_1 \\ 1 - \gamma_{p+1} + \gamma_1, \dots, 1 - \gamma_n + \gamma_1 \end{matrix}; z \right) \end{aligned}$$

6. SERIES EXPANSIONS

6.1. Evaluation of the integrals. Ultimately, we want to compare the analytic continuation of the integral representation with the series solutions obtained from the Frobenius method. To achieve this we have to evaluate the various integrals that we obtained in Section 5. We formulate the results as corollaries.

For this purpose we define the numbers $A^{(n)}(k)$ by taking the residue of the functions $\tilde{A}^{(n)}(u)$ given in Proposition 4.8 at $u = k, k \in \mathbb{N}$:

$$A^{(n)}(k) := \text{Res}_{u=k} \tilde{A}^{(n)}(u).$$

Of course, the internal integrals of $A^{(n)}(u)$ for $n > 3$ can be evaluated with the residue theorem. For later purposes we give the result for $n = 3$ and $n = 4$ obtained from (4.3) and (4.4), respectively:

$$(6.1) \quad A^{(3)}(k) = \frac{\Gamma(b_2 - a_3 + k) \Gamma(b_1 - a_3 + k)}{\Gamma(b_2 - a_3) \Gamma(b_1 - a_3) \Gamma(k+1)}$$

$$\begin{aligned} (6.2) \quad A^{(4)}(k) &= \frac{\Gamma(b_3 + b_2 - a_4 - a_3 + k)}{\Gamma(b_3 - a_4) \Gamma(b_2 - a_4) \Gamma(b_1 - a_3)} \\ &\quad \cdot \sum_{k_2=0}^k \frac{\Gamma(b_1 - a_3 + k - k_2) \Gamma(b_3 - a_4 + k_2) \Gamma(b_2 - a_4 + k_2)}{\Gamma(k - k_2 + 1) \Gamma(b_3 + b_2 - a_4 - a_3 + k_2) \Gamma(k_2 + 1)} \end{aligned}$$

As a corollary to Proposition 5.2 we obtain the following series expansions in powers of $1 - z$ of ${}_nF_{n-1}(z)$ due to Bühring [2]. They strongly depend on the integrality of the number c in (4.2).

Corollary 6.1. *If $c \notin \mathbb{Z}$ then*

$$\begin{aligned} & \frac{\Gamma(a_1) \cdots \Gamma(a_n)}{\Gamma(b_1) \cdots \Gamma(b_{n-1})} {}_nF_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) \\ &= \sum_{m=0}^{\infty} g_m(0)(1-z)^m + (1-z)^c \sum_{m=0}^{\infty} g_m(c)(1-z)^m \end{aligned}$$

where

$$\begin{aligned} g_m(\ell) &= (-1)^m \frac{\Gamma(a_1 + \ell + m) \Gamma(a_2 + \ell + m) \Gamma(c - 2\ell - m)}{\Gamma(c + a_1) \Gamma(c + a_2) \Gamma(m + 1)} \\ &\quad \cdot \sum_{k=0}^{\infty} \frac{(c - \ell - m)_k}{(c + a_1)_k (c + a_2)_k} A^{(n)}(k) \end{aligned}$$

The series in $g_m(\ell)$ terminates when $\ell = c$, while for $\ell = 0$ we need the conditions $\Re(a_j + m) > 0$, $j = 3, \dots, n$, for convergence.

Proof. We evaluate the integral on the right-hand side of Proposition 5.2.

$$\int \frac{dt}{2\pi i} \int \frac{du}{2\pi i} \tilde{A}^{(n)}(t) \frac{\Gamma(a_1 + u) \Gamma(a_2 + u) \Gamma(c + t - u) \Gamma(-u)}{\Gamma(c + a_2 + t) \Gamma(c + a_1 + t)} (1 - z)^u$$

Since c is not an integer, we first carry out the integral over t , then the integral over u . The integral over t is simply obtained by closing the contour to the right

$$\int \frac{dt}{2\pi i} \tilde{A}^{(n)}(t) \frac{\Gamma(c + t - u)}{\Gamma(c + a_2 + t) \Gamma(c + a_1 + t)} = \sum_{k \geq 0} A_n(k) \frac{\Gamma(c + k - u)}{\Gamma(c + a_2 + k) \Gamma(c + a_1 + k)}$$

The remaining integral over u is a G_2 with lower parameters $c + k$ and 0 which, by assumption, are not in resonance. Hence we can apply Lemma 2.3(1) and the reflection formula for the Gamma function.

$$\begin{aligned} & \int \frac{du}{2\pi i} \Gamma(a_1 + u) \Gamma(a_2 + u) \Gamma(c + k - u) \Gamma(-u) (1 - z)^u \\ &= \Gamma(c + k) \Gamma(1 - c - k) \int_{-i\infty}^{i\infty} \frac{du}{2\pi i} \frac{\Gamma(a_1 + u) \Gamma(a_2 + u) \Gamma(-u)}{\Gamma(1 - c - k + u)} (z - 1)^u \\ &\quad + (1 - z)^{c+k} \Gamma(-c - k) \Gamma(1 + c + k) \\ &\quad \cdot \int \frac{du}{2\pi i} \frac{\Gamma(c + a_1 + k + u) \Gamma(c + a_2 + k + u) \Gamma(-u)}{\Gamma(1 + c + k + u)} (z - 1)^u \end{aligned}$$

Shifting the integration variable in the second integral and closing the contour to the right in both integrals, this expression becomes

$$\begin{aligned} & \Gamma(c + k) \Gamma(1 - c - k) \sum_{m=0}^{\infty} \frac{\Gamma(a_1 + m) \Gamma(a_2 + m)}{\Gamma(1 - c - k + m) \Gamma(m + 1)} (1 - z)^m \\ &+ (1 - z)^c \Gamma(-c - k) \Gamma(1 + c + k) \sum_{m=0}^{\infty} \frac{\Gamma(c + a_1 + m) \Gamma(c + a_2 + m)}{\Gamma(1 + c + m) \Gamma(m - k + 1)} (1 - z)^m \end{aligned}$$

Hence, combining everything we find

$$\begin{aligned} & \frac{\Gamma(a_1) \cdots \Gamma(a_n)}{\Gamma(b_1) \cdots \Gamma(b_{n-1})} {}_nF_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) \\ &= \sum_{k=0}^{\infty} \frac{A^{(n)}(k) \Gamma(c+k) \Gamma(1-c-k)}{\Gamma(c+a_2+k) \Gamma(c+a_1+k)} \sum_{m=0}^{\infty} \frac{\Gamma(a_1+m) \Gamma(a_2+m)}{\Gamma(1-c-k+m) \Gamma(m+1)} (1-z)^m \\ & \quad + \frac{A^{(n)}(k) \Gamma(-c-k) \Gamma(1+c+k)}{\Gamma(c+a_2+k) \Gamma(c+a_1+k)} (1-z)^c \sum_{m=0}^{\infty} \frac{\Gamma(c+a_1+m) \Gamma(c+a_2+m)}{\Gamma(1+c+m) \Gamma(m-k+1)} (1-z)^m \end{aligned}$$

We massage this expression by using $\Gamma(\alpha+\ell)\Gamma(1-\alpha-\ell) = (-1)^\ell \Gamma(\alpha)\Gamma(1-\alpha)$ for $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{Z}$ three times: With $\alpha = c+k$, $\ell = m$ in the first summand, and with $\alpha = -c$, $\ell = m+k$, as well as $\alpha = 1+m$, $\ell = -k$ in the second summand. The convergence of the sums over m for $\Re(a_j+m) > 0$, $j = 3, \dots, n$, allows us to interchange the sums. This yields

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{A^{(n)}(k)}{\Gamma(c+a_2+k) \Gamma(c+a_1+k)} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(a_1+m) \Gamma(a_2+m) \Gamma(c+k-m)}{\Gamma(m+1)} (1-z)^m \\ & \quad + \frac{A^{(n)}(k)}{\Gamma(c+a_2+k) \Gamma(c+a_1+k)} \\ & \quad \cdot (1-z)^c \sum_{m=0}^{\infty} (-1)^{m+k} \frac{\Gamma(c+a_1+m) \Gamma(c+a_2+m) \Gamma(-c-m)}{\Gamma(m-k+1)} (1-z)^m \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(a_1+m) \Gamma(a_2+m)}{\Gamma(m+1)} \sum_{k=0}^{\infty} \frac{A^{(n)}(k) \Gamma(c+k-m)}{\Gamma(c+a_2+k) \Gamma(c+a_1+k)} (1-z)^m \\ & \quad + (1-z)^c \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(-m-c) \Gamma(c+a_1+m) \Gamma(c+a_2+m)}{\Gamma(1+m) \Gamma(-m)} \\ & \quad \cdot \sum_{k=0}^{\infty} (-1)^k \frac{A^{(n)}(k) \Gamma(-m+k)}{\Gamma(c+a_2+k) \Gamma(c+a_1+k)} (1-z)^m \end{aligned}$$

Rewriting this expression in terms of Pochhammer symbols yields the claim. \square

For the applications in Section 7 we need the case $n = 3$ explicitly. In particular,

$$(6.3) \quad g_0(c) = \Gamma(-c).$$

in agreement with Nørlund's analysis [14]. Since the indices of the hypergeometric differential equation at $z = 1$ are $0, 1$ and $c = b_1 + b_2 - a_1 - a_2 - a_3$, it is sufficient to determine the coefficients $g_j(0)$ for $j = 0, 1$ besides $g_0(c)$. With (6.1) we find

$$\begin{aligned} (6.4) \quad g_0(0) &= \frac{\Gamma(a_1) \Gamma(a_2) \Gamma(c)}{\Gamma(c+a_1) \Gamma(c+a_2)} \sum_{k=0}^{\infty} \frac{(c)_k (b_2-a_3)_k (b_1-a_3)_k}{(a_1+c)_k (a_2+c)_k k!} \\ &= \frac{\Gamma(a_1) \Gamma(a_2) \Gamma(c)}{\Gamma(c+a_1) \Gamma(c+a_2)} {}_3F_2 \left(\begin{matrix} c, b_2-a_3, b_1-a_3 \\ a_1+c, a_2+c \end{matrix}; 1 \right) \end{aligned}$$

and

$$\begin{aligned} (6.5) \quad g_1(0) &= -\frac{\Gamma(a_1+1) \Gamma(a_2+1) \Gamma(c-1)}{\Gamma(c+a_1) \Gamma(c+a_2)} \sum_{k=0}^{\infty} \frac{(c-1)_k (b_2-a_3)_k (b_1-a_3)_k}{(a_1+c)_k (a_2+c)_k k!} \\ &= -\frac{\Gamma(a_1+1) \Gamma(a_2+1) \Gamma(c-1)}{\Gamma(c+a_1) \Gamma(c+a_2)} {}_3F_2 \left(\begin{matrix} c-1, b_2-a_3, b_1-a_3 \\ a_1+c, a_2+c \end{matrix}; 1 \right) \end{aligned}$$

If c is a nonnegative integer, Proposition 5.2 yields an entirely different series expansion, again due to [2].

Corollary 6.2. *If $c = c_0 \in \mathbb{Z}_{\geq 0}$ then*

$$\begin{aligned} & \frac{\Gamma(a_1) \cdots \Gamma(a_n)}{\Gamma(b_1) \cdots \Gamma(b_{n-1})} {}_nF_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) \\ &= \sum_{m=0}^{c_0-1} l_m (1-z)^m + (1-z)^{c_0} \sum_{m=0}^{\infty} (w_m + q_m \log(1-z)) (1-z)^m \end{aligned}$$

with

$$\begin{aligned} l_m &= g_m(0)|_{c=c_0} = (-1)^m \frac{\Gamma(a_1+m)\Gamma(a_2+m)\Gamma(c_0-m)}{\Gamma(c_0+a_1)\Gamma(c_0+a_2)\Gamma(m+1)} \sum_{k=0}^{\infty} \frac{(c_0-m)_k}{(a_1+c_0)_k(a_2+c_0)_k} A^{(n)}(k) \\ q_m &= g_m(c)|_{c=c_0} = (-1)^{c_0+1} \frac{(a_1+c_0)_m(a_2+c_0)_m}{\Gamma(c_0+m+1)\Gamma(m+1)} \sum_{k=0}^m \frac{(-m)_k}{(a_1+c_0)_k(a_2+c_0)_k} A^{(n)}(k) \\ w_m &= (-1)^{c_0} \frac{(a_1+c_0)_m(a_2+c_0)_m}{\Gamma(c_0+m+1)\Gamma(m+1)} \sum_{k=0}^m \frac{(-m)_k}{(a_1+c_0)_k(a_2+c_0)_k} A^{(n)}(k) \\ &\quad \cdot (\psi(1+m-k) + \psi(1+c_0+m) - \psi(a_1+c_0+m) - \psi(a_2+c_0+m)) \\ &\quad + (-1)^{c_0+m} \frac{(a_1+c_0)_m(a_2+c_0)_m}{\Gamma(c_0+m+1)} \sum_{k=m+1}^{\infty} \frac{\Gamma(k-m)}{(a_1+c_0)_k(a_2+c_0)_k} A^{(n)}(k) \end{aligned}$$

The convergence of the series in l_m requires the conditions $\Re(a_j + m) > 0$, $j = 3, \dots, n$, while the convergence of the series in w_m requires the conditions $\Re(c_0 + a_j + m) > 0$, $j = 3, \dots, n$.

Proof. We begin as in the proof of Corollary 6.1 and evaluate the t integral in Proposition 5.2. The remaining integral over u is a G_2 with lower parameters $c+k$ and 0 which, by assumption, are now in resonance. Instead of applying Lemma 2.3(1) we have to evaluate the residue integral explicitly by closing the contour to the right and picking up the double poles. This is a lengthy but straightforward computation whose result can be found e.g. in [6] or in [15]. It reads in our case

$$\begin{aligned} & (-1)^{c_0+k} \int \frac{du}{2\pi i} \Gamma(a_1+u)\Gamma(a_2+u)\Gamma(c_0+k-u)\Gamma(-u)(1-z)^u \\ &= -\log(1-z)(1-z)^{c_0+k} \sum_{m \geq 0} \frac{\Gamma(a_1+c_0+k+m)\Gamma(a_2+c_0+k+m)}{\Gamma(1+c_0+k+m)\Gamma(m+1)} (1-z)^m \\ &\quad - (1-z)^{c_0+k} \sum_{m=0}^{\infty} \frac{\Gamma(a_1+c_0+k+m)\Gamma(a_2+c_0+k+m)}{\Gamma(1+c_0+k+m)\Gamma(m+1)} \\ &\quad \cdot (\psi(a_1+c_0+k+m) + \psi(a_2+c_0+k+m) - \psi(1+c_0+k+m) - \psi(1+m)) (1-z)^m \\ &\quad + (-1)^{c_0+k} \Gamma(c_0+k)\Gamma(1-c_0-k) \sum_{m=0}^{c_0+k-1} \frac{\Gamma(a_1+m)\Gamma(a_2+m)}{\Gamma(1-c_0-k+m)\Gamma(m+1)} (1-z)^m \end{aligned}$$

where $\psi(z) = (\log \Gamma(z))'$ is the Digamma function. Hence, combining everything we find

$$\begin{aligned} & \frac{\Gamma(a_1) \cdots \Gamma(a_n)}{\Gamma(b_1) \cdots \Gamma(b_{n-1})} {}_nF_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; z \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{c_0+k} A^{(n)}(k)}{\Gamma(c_0+a_2+k)\Gamma(c_0+a_1+k)} \left(-\log(1-z)(1-z)^{c_0+k} \right. \\ &\quad \cdot \sum_{m=0}^{\infty} \frac{\Gamma(a_1+c_0+k+m)\Gamma(a_2+c_0+k+m)}{\Gamma(1+c_0+k+m)\Gamma(m+1)} (1-z)^m \end{aligned}$$

$$\begin{aligned}
& - (1-z)^{c_0+k} \sum_{m=0}^{\infty} \frac{\Gamma(a_1+c_0+k+m)\Gamma(a_2+c_0+k+m)}{\Gamma(1+c_0+k+m)\Gamma(m+1)} \\
& \cdot (\psi(a_1+c_0+k+m) + \psi(a_2+c_0+k+m) - \psi(1+c_0+k+m) - \psi(1+m)) (1-z)^m \\
& + (-1)^{c_0+k} \Gamma(c_0+k)\Gamma(1-c_0-k) \sum_{m=0}^{c_0+k-1} \frac{\Gamma(a_1+m)\Gamma(a_2+m)}{\Gamma(1-c_0-k+m)\Gamma(m+1)} (1-z)^m \Big)
\end{aligned}$$

For the first two summands in the parenthesis a standard argument in the theory of hypergeometric functions shows that the infinite sums over m converge for $\Re(a_j + m) > 0$, $j = 3, \dots, n$, hence we can interchange the sums over m and k . In the third summand we split the sum over m into a sum from 0 to $c_0 - 1$ and a sum over the remaining values of m . In the latter we shift the summation index. Then this expression on the right hand side becomes

$$\begin{aligned}
& (1-z)^{c_0} \log(1-z) \sum_{m=0}^{\infty} (-1)^{c_0+1} \frac{\Gamma(a_1+c_0+m)\Gamma(a_2+c_0+m)}{\Gamma(1+c_0+m)} \\
& \cdot \sum_{k=0}^m \frac{(-1)^k A^{(n)}(k)}{\Gamma(a_1+c_0+k)\Gamma(a_2+c_0+k)\Gamma(1+m-k)} (1-z)^m \\
& + (1-z)^{c_0} \sum_{m=0}^{\infty} (-1)^{c_0+1} \frac{\Gamma(a_1+c_0+m)\Gamma(a_2+c_0+m)}{\Gamma(1+c_0+m)} \\
& \cdot \sum_{k=0}^m \frac{(-1)^k A^{(n)}(k)}{\Gamma(a_1+c_0+k)\Gamma(a_2+c_0+k)\Gamma(1+m-k)} \\
& \cdot (\psi(a_1+c_0+m) + \psi(a_2+c_0+m) - \psi(1+c_0+m) - \psi(1+m-k)) (1-z)^m \\
& + \sum_{k=0}^{\infty} \frac{A^{(n)}(k)\Gamma(c_0+k)\Gamma(1-c_0-k)}{\Gamma(a_1+c_0+k)\Gamma(a_2+c_0+k)} \sum_{m=0}^{c_0-1} \frac{\Gamma(a_1+m)\Gamma(a_2+m)}{\Gamma(1-c_0-k+m)\Gamma(m+1)} (1-z)^m \\
& + (1-z)^{c_0} \sum_{k=0}^{\infty} \frac{A^{(n)}(k)\Gamma(c_0+k)\Gamma(1-c_0-k)}{\Gamma(a_1+c_0+k)\Gamma(a_2+c_0+k)} \sum_{m=0}^k \frac{\Gamma(a_1+c_0+m)\Gamma(a_2+c_0+m)}{\Gamma(1-k+m)\Gamma(1+c_0+m)} (1-z)^m
\end{aligned}$$

Again, we massage this expression by using $\Gamma(\alpha+\ell)\Gamma(1-\alpha-\ell) = (-1)^\ell \Gamma(\alpha)\Gamma(1-\alpha)$ for $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{Z}$ four times: With $\alpha = 1+m$, $\ell = -k$ in the first and second summand, with $\alpha = c_0+k$, $\ell = m$ in the third summand, and with $\alpha = c_0+k$, $\ell = -m-c_0$. Furthermore, if we require that $\Re(c_0+a_j+m) > 0$, $j = 3, \dots, n$, we can interchange the sums over m and k in the fourth summand. This yields

$$\begin{aligned}
& (1-z)^{c_0} \log(1-z) \sum_{m=0}^{\infty} (-1)^{c_0+1} \frac{\Gamma(a_1+c_0+m)\Gamma(a_2+c_0+m)}{\Gamma(1+c_0+m)\Gamma(m+1)} \\
& \cdot \sum_{k=0}^m \frac{A^{(n)}(k)\Gamma(k-m)}{\Gamma(a_1+c_0+k)\Gamma(a_2+c_0+k)\Gamma(-m)} (1-z)^m \\
& + (1-z)^{c_0} \sum_{m=0}^{\infty} (-1)^{c_0+1} \frac{\Gamma(a_1+c_0+m)\Gamma(a_2+c_0+m)}{\Gamma(1+c_0+m)\Gamma(m+1)} \\
& \cdot \sum_{k=0}^m \frac{A^{(n)}(k)\Gamma(k-m)}{\Gamma(a_1+c_0+k)\Gamma(a_2+c_0+k)\Gamma(-m)} \\
& \cdot (\psi(a_1+c_0+m) + \psi(a_2+c_0+m) - \psi(1+c_0+m) - \psi(1+m-k)) (1-z)^m
\end{aligned}$$

$$\begin{aligned}
& + \sum_{m=0}^{c_0-1} (-1)^m \frac{\Gamma(a_1+m)\Gamma(a_2+m)}{\Gamma(m+1)} \sum_{k=0}^{\infty} \frac{A^{(n)}(k)\Gamma(c_0+k-m)}{\Gamma(a_1+c_0+k)\Gamma(a_2+c_0+k)} (1-z)^m \\
& + (1-z)^{c_0} \sum_{m=0}^{\infty} (-1)^{m+c_0} \frac{\Gamma(a_1+c_0+m)\Gamma(a_2+c_0+m)}{\Gamma(1+c_0+m)} \\
& \cdot \sum_{k=m+1}^{\infty} \frac{A^{(n)}(k)\Gamma(k-m)}{\Gamma(a_1+c_0+k)\Gamma(a_2+c_0+k)} (1-z)^m
\end{aligned}$$

Rewriting this expression in terms of Pochhammer symbols yields the claim. \square

For the applications in Section 7 we need the case $n = 4$ and $c_0 = 1$ explicitly. The four coefficients to be determined are

$$(6.6) \quad l_0 = \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(1+a_1)\Gamma(1+a_2)} \sum_{k=0}^{\infty} \frac{\Gamma(1+k)}{(a_1+1)_k(a_2+1)_k} A^{(4)}(k)$$

$$(6.7) \quad w_0 = A^{(4)}(0)$$

$$(6.8) \quad q_0 = A^{(4)}(0) (\psi(a_1+1) + \psi(a_2+1) - \psi(1) - \psi(2)) - \sum_{k=1}^{\infty} \frac{\Gamma(k)}{(a_1+1)_k(a_2+1)_k} A^{(4)}(k)$$

$$(6.9) \quad q_1 = \frac{(a_1+1)(a_2+1)}{2} \left(\sum_{k=0}^1 \frac{(-1)_k}{(a_1+1)_k(a_2+1)_k} A^{(4)}(k) \cdot (\psi(a_1+2) + \psi(a_2+2) - \psi(2-k) - \psi(3)) + \sum_{k=2}^{\infty} \frac{\Gamma(k-1)}{(a_1+1)_k(a_2+1)_k} A^{(4)}(k) \right)$$

with

$$(6.10) \quad A^{(4)}(k) = \frac{\Gamma(b_3+b_2-a_4-a_3+k)\Gamma(b_1-a_3+k)}{\Gamma(b_3+b_2-a_4-a_3)\Gamma(b_1-a_3)\Gamma(1+k)} \cdot {}_3F_2 \left(\begin{matrix} b_3-a_4, b_2-a_4, -k \\ b_3+b_2-a_4-a_3, 1+a_3-b_1-k \end{matrix}; 1 \right)$$

Remark 6.3. The analogous result for c a nonpositive integer can be derived in a parallel manner. The evaluation of the u integral now yields a different result, see again [6] or [15]. The final result for the analytic continuation of ${}_nF_{n-1}$ in this case can be found in [2].

As a corollary to Proposition 5.3 we obtain the following series expansion in powers of $1-z$ of $G_p(z)$. We start with an easy lemma:

Lemma 6.4. *For any $1 < p \leq q \leq n$, if $|z-1| < 1$, then*

$$G_p \left(\begin{matrix} \alpha_1, \dots, \alpha_n \\ \gamma_1, \dots, \gamma_n \end{matrix}; z \right) = z^{\gamma_q} \sum_{m=0}^{\infty} \frac{1}{m!} G_p \left(\begin{matrix} \alpha_1, \dots, \alpha_n \\ \gamma_1, \dots, \gamma_{q-1}, \gamma_q+m, \gamma_{q+1}, \dots, \gamma_n \end{matrix}; 1 \right) (1-z)^m$$

Proof. This almost immediately follows from the definition of $G_p(z)$ in terms of the Meijer G-function in Section 2.2. In fact,

$$\begin{aligned}
& G_p \left(\begin{matrix} \alpha_1, \dots, \alpha_n \\ \gamma_1, \dots, \gamma_n \end{matrix}; z \right) \\
& = z^{\gamma_q} \sum_{m=0}^{\infty} \frac{1}{m!} \left. \frac{d^m (z^{-\gamma_q} G_p(z))}{dz^m} \right|_{z=1} (z-1)^m
\end{aligned}$$

$$\begin{aligned}
&= z^{\gamma_q} \sum_{m=0}^{\infty} \frac{1}{m!} \int \frac{dt}{2\pi i} e^{i\pi t(p-2)} \prod_{j=1}^n \frac{\Gamma(\alpha_j + t)}{\Gamma(1 - \gamma_j + t)} \prod_{h=1}^p \Gamma(\gamma_h - t) \Gamma(1 - \gamma_h + t) \\
&\quad \cdot \left. \frac{d^m(z^{t-\gamma_q})}{dz^m} \right|_{z=1} (z-1)^m \\
&= z^{\gamma_q} \sum_{m=0}^{\infty} \frac{1}{m!} \int \frac{dt}{2\pi i} e^{i\pi t(p-2)} \prod_{j=1}^n \frac{\Gamma(\alpha_j + t)}{\Gamma(1 - \gamma_j + t)} \prod_{h=1}^p \Gamma(\gamma_h - t) \Gamma(1 - \gamma_h + t) \\
&\quad \cdot \frac{\Gamma(\gamma_q - t + m)}{\Gamma(\gamma_q - t)} (1-z)^m
\end{aligned}$$

For $p = 2$ this is due to [14]. \square

Theorem 6.5. *For any $2 < p \leq q \leq n$, if $|z-1| < 1$, $\Re\beta_n > \Re\beta_p$, $\Re(\alpha_s + \gamma_j) > 0$, $j = 1, \dots, p$, $s = p+1, \dots, n$, $\alpha_p + \gamma_p, \alpha_s + \gamma_{s+1} \notin \mathbb{Z}_{\leq 0}$, $s = 2, \dots, p-1$ then*

$$\begin{aligned}
G_p(z) &= \sum_{m=0}^{\infty} \Gamma(\alpha_1 + \gamma_2) \int \frac{dv}{2\pi i} e^{-i\pi v} \Gamma(\alpha_1 + \gamma_1 + v) \Gamma(-v) \\
&\quad \cdot \int \frac{ds}{2\pi i} \frac{B_{p,m}(s)}{\Gamma(m+1)} \frac{\Gamma(\gamma_2 - s) \Gamma(\gamma_1 + v - s)}{\Gamma(\alpha_1 + \gamma_1 + \gamma_2 + v - s)} \\
&\quad \cdot \int \frac{du}{2\pi i} e^{-i\pi u} \frac{\Gamma(-v + u) \Gamma(-u)}{\Gamma(-v)} \prod_{s=p+1}^n \frac{\Gamma(\alpha_s + \gamma_1 + u)}{\Gamma(1 - \gamma_s + \gamma_1 + u)} (1-z)^m
\end{aligned}$$

where $B_{p,m}(s) = B_p(s)|_{\gamma_p \rightarrow \gamma_p + m}$ with $B_p(s)$ as in Proposition 4.10. If $p = 2$ then

$$\begin{aligned}
G_2(z) &= \sum_{m=0}^{\infty} \frac{\Gamma(\alpha_1 + \gamma_2 + m) \Gamma(\alpha_2 + \gamma_2 + m)}{\Gamma(m+1)} \\
&\quad \cdot \int \frac{dv}{2\pi i} e^{-i\pi v} \frac{\Gamma(\alpha_1 + \gamma_1 + v) \Gamma(\alpha_2 + \gamma_1 + v) \Gamma(-v)}{\Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + m + v)} \\
&\quad \cdot \int \frac{du}{2\pi i} e^{-i\pi u} \frac{\Gamma(-v + u) \Gamma(-u)}{\Gamma(-v)} \prod_{s=3}^n \frac{\Gamma(\alpha_s + \gamma_1 + u)}{\Gamma(1 - \gamma_s + \gamma_1 + u)} (1-z)^m
\end{aligned}$$

Proof. This follows from Proposition 5.3 and Lemma 6.4. The s integral can be expressed in terms multiple integrals of hypergeometric functions. By Remark 5.4 the u integral is the evaluation at $z = 1$ of a hypergeometric function which requires a convergence condition that follows from the following asymptotic expansion [14].

$${}_{n+1}F_n \left(\begin{matrix} a_1, \dots, a_n, -x \\ b_1, \dots, b_n \end{matrix}; z \right) \sim \sum_{i=1}^n C_i x^{-a_i} (\log x)^{r_i},$$

for some constants C_i , nonnegative integers r_i and $|z-1| < 1$. \square

Remark 6.6. By using the residue theorem, we can give an integral representation for $G_p(z)$ as a function of $1-z$.

$$\begin{aligned}
G_p(z) &= \Gamma(\alpha_1 + \gamma_2) \int \frac{dt}{2\pi i} e^{i\pi(t+1)} \Gamma(-t) \int \frac{dv}{2\pi i} e^{-i\pi v} \Gamma(\alpha_1 + \gamma_1 + v) \Gamma(-v) \\
&\quad \cdot \int \frac{ds}{2\pi i} B_p(s, t) \frac{\Gamma(\gamma_2 - s) \Gamma(\gamma_1 + v - s)}{\Gamma(\alpha_1 + \gamma_1 + \gamma_2 + v - s)} \\
&\quad \cdot \int \frac{du}{2\pi i} e^{-i\pi u} \frac{\Gamma(-v + u) \Gamma(-u)}{\Gamma(-v)} \prod_{s=p+1}^n \frac{\Gamma(\alpha_s + \gamma_1 + u)}{\Gamma(1 - \gamma_s + \gamma_1 + u)} (1-z)^t
\end{aligned}$$

where $B_p(s, t) = B_p(s)|_{\gamma_p \rightarrow \gamma_p + t}$.

6.2. Examples. For our applications in Section 7 we need a few cases explicitly.

Example 6.7. For $n = 3, p = 2$ we have

$$(6.11) \quad G_2(z) = z^{\gamma_1} \sum_{m=0}^{\infty} h_m (1-z)^m$$

with

$$h_m = \frac{\Gamma(\alpha_3 + \gamma_1)\Gamma(\alpha_1 + \gamma_2 + m)\Gamma(\alpha_2 + \gamma_2 + m)}{\Gamma(1 - \gamma_3 + \gamma_1)\Gamma(m+1)} \cdot \int \frac{dv}{2\pi i} e^{-i\pi v} \frac{\Gamma(\alpha_1 + \gamma_1 + v)\Gamma(\alpha_2 + \gamma_1 + v)\Gamma(-v)}{\Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + m + v)} {}_2F_1 \left(\begin{matrix} -v, \alpha_3 + \gamma_1 \\ 1 - \gamma_3 + \gamma_1 \end{matrix}; 1 \right)$$

By applying Gauss' formula (3.1) this becomes

$$(6.12) \quad \begin{aligned} h_m &= \frac{\Gamma(\alpha_3 + \gamma_1)\Gamma(\alpha_1 + \gamma_2 + m)\Gamma(\alpha_2 + \gamma_2 + m)}{\Gamma(1 - \alpha_3 - \gamma_3)\Gamma(m+1)} \\ &\cdot \int \frac{dv}{2\pi i} e^{-i\pi v} \frac{\Gamma(\alpha_1 + \gamma_1 + v)\Gamma(\alpha_2 + \gamma_1 + v)\Gamma(1 - \alpha_3 - \gamma_3 + v)\Gamma(-v)}{\Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + m + v)\Gamma(1 + \gamma_1 - \gamma_3 + v)} \\ &= \frac{\Gamma(\alpha_1 + \gamma_1)\Gamma(\alpha_2 + \gamma_1)\Gamma(\alpha_3 + \gamma_1)\Gamma(\alpha_1 + \gamma_2 + m)\Gamma(\alpha_2 + \gamma_2 + m)}{\Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + m)\Gamma(1 + \gamma_1 - \gamma_3)\Gamma(m+1)} \\ &\cdot {}_3F_2 \left(\begin{matrix} \alpha_1 + \gamma_1, \alpha_2 + \gamma_1, 1 - \alpha_3 - \gamma_3 \\ \alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + m, 1 + \gamma_1 - \gamma_3 \end{matrix}; 1 \right) \end{aligned}$$

Example 6.8. For $n = 4, p = 2$ we have

$$(6.13) \quad G_2(z) = z^{\gamma_1} \sum_{m=0}^{\infty} h_m (1-z)^m$$

with

$$h_m = \frac{\Gamma(\alpha_3 + \gamma_1)\Gamma(\alpha_4 + \gamma_1)\Gamma(\alpha_1 + \gamma_2 + m)\Gamma(\alpha_2 + \gamma_2 + m)}{\Gamma(1 - \gamma_3 + \gamma_1)\Gamma(1 - \gamma_4 + \gamma_1)\Gamma(m+1)} \cdot \int \frac{dv}{2\pi i} e^{-i\pi v} \frac{\Gamma(\alpha_1 + \gamma_1 + v)\Gamma(\alpha_2 + \gamma_1 + v)\Gamma(-v)}{\Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + m + v)} \cdot {}_3F_2 \left(\begin{matrix} -v, \alpha_3 + \gamma_1, \alpha_4 + \gamma_1 \\ 1 - \gamma_3 + \gamma_1, 1 - \gamma_4 + \gamma_1 \end{matrix}; 1 \right)$$

We evaluate the v integral by closing the contour to the right and obtain

$$(6.14) \quad \begin{aligned} h_m &= \frac{\Gamma(\alpha_3 + \gamma_1)\Gamma(\alpha_4 + \gamma_1)\Gamma(\alpha_1 + \gamma_2 + m)\Gamma(\alpha_2 + \gamma_2 + m)}{\Gamma(1 - \gamma_3 + \gamma_1)\Gamma(1 - \gamma_4 + \gamma_1)\Gamma(m+1)} \\ &\cdot \sum_{\ell=0}^{\infty} \frac{\Gamma(\alpha_1 + \gamma_1 + \ell)\Gamma(\alpha_2 + \gamma_1 + \ell)}{\Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + m + \ell)\Gamma(\ell+1)} {}_3F_2 \left(\begin{matrix} -\ell, \alpha_3 + \gamma_1, \alpha_4 + \gamma_1 \\ 1 - \gamma_3 + \gamma_1, 1 - \gamma_4 + \gamma_1 \end{matrix}; 1 \right) \end{aligned}$$

For the last example we will need a theorem of Slater which we state here as lemma

Lemma 6.9. Let $p, q, r, s \in \mathbb{Z}_{\geq 0}$, $a_1, \dots, a_p, b_1, \dots, b_q, c_1, \dots, c_r, d_1, \dots, d_s \in \mathbb{C}$ and

$$I(z) = \int \frac{dt}{2\pi i} \frac{\prod_{i=1}^p \Gamma(a_i + t) \prod_{j=1}^q \Gamma(b_j - t)}{\prod_{k=1}^r \Gamma(c_k + t) \prod_{\ell=1}^s \Gamma(d_\ell - t)} z^t$$

Then

$$I(z) = \sum_{m=1}^q z^{b_m} \frac{\prod_{i=1}^p \Gamma(a_i + b_m) \prod_{j=1, j \neq m}^q \Gamma(b_j - b_m)}{\prod_{k=1}^r \Gamma(c_k + b_m) \prod_{\ell=1}^s \Gamma(d_\ell - b_m)} \\ \cdot {}_{p+s}F_{q+r-1} \left(\begin{matrix} a_1 + b_m, \dots, a_p + b_m, 1 + b_m - d_1, \dots, 1 + b_m - d_s \\ c_1 + b_m, \dots, c_r + b_m, 1 + b_m - b_1, \widehat{\dots}, 1 + b_m - b_q \end{matrix}; (-1)^{q+s} z \right)$$

provided that $\lambda - \mu \notin \mathbb{Z}$ for all $\lambda, \mu \in \{b_1, \dots, b_q, c_1, \dots, c_r\}$ and

$$\frac{1}{2}\pi|p + q - r - s| > |\arg z|, \quad q + r \geq p + s, \quad |z| < 1,$$

Also,

$$I(z) = \sum_{m=1}^p z^{-a_m} \frac{\prod_{i=1, i \neq m}^p \Gamma(a_i - a_m) \prod_{j=1}^q \Gamma(b_j + a_m)}{\prod_{k=1}^r \Gamma(c_k - a_m) \prod_{\ell=1}^s \Gamma(d_\ell + a_m)} \\ \cdot {}_{q+r}F_{p+s-1} \left(\begin{matrix} b_1 + a_m, \dots, b_q + a_m, 1 + a_m - c_1, \dots, 1 + a_m - c_r \\ d_1 + a_m, \dots, d_s + a_m, 1 + a_m - a_1, \widehat{\dots}, 1 + a_m - a_p \end{matrix}; \frac{(-1)^{p+r}}{z} \right)$$

provided that $\lambda - \mu \notin \mathbb{Z}$ for all $\lambda, \mu \in \{a_1, \dots, a_p, d_1, \dots, d_s\}$ and

$$\frac{1}{2}\pi|p + q - r - s| > |\arg z|, \quad p + s \geq q + r, \quad |z| < 1.$$

In addition, both formulas are valid for $z = 1$ if furthermore

$$\Re \left(\sum_{k=1}^r c_k + \sum_{\ell=1}^s d_\ell - \sum_{i=1}^p a_i - \sum_{j=1}^q b_j \right) > 0.$$

Proof. See [16]. In fact, Lemma 2.3(1) is special case of this lemma. \square

Example 6.10. Finally, for $n = 4, p = 3$ we have

$$(6.15) \quad G_3(z) = z^{\gamma_1} \sum_{m=0}^{\infty} k_m (1 - z)^m$$

with

$$(6.16) \quad k_m = \frac{\Gamma(\alpha_1 + \gamma_2) \Gamma(\alpha_2 + \gamma_3 + m) \Gamma(\alpha_3 + \gamma_3 + m) \Gamma(\alpha_4 + \gamma_1)}{\Gamma(1 - \alpha_4 - \gamma_4) \Gamma(m + 1)} \\ \cdot \int \frac{dv}{2\pi i} e^{-i\pi v} \frac{\Gamma(\alpha_1 + \gamma_1 + v) \Gamma(1 - \alpha_4 - \gamma_4 + v) \Gamma(-v)}{\Gamma(1 - \gamma_4 + \gamma_1 + v)} \\ \cdot \int \frac{ds}{2\pi i} e^{-i\pi s} \frac{\Gamma(\alpha_2 + s) \Gamma(\alpha_3 + s) \Gamma(\gamma_2 - s) \Gamma(\gamma_1 + v - s)}{\Gamma(\alpha_2 + \alpha_3 + \gamma_3 + m + s) \Gamma(\alpha_1 + \gamma_1 + \gamma_2 + v - s)}$$

where we have used (4.5) and Gauss' formula (3.1). The two integrals can be evaluated with the help of Lemma 6.9. In particular, if $\Re(\alpha_1 + \gamma_3 + m) > 0$ and if $\alpha_1, \dots, \alpha_3, \gamma_1, \dots, \gamma_3$ are such that there are only simple poles, we obtain for the s integral in

$$\int \frac{ds}{2\pi i} e^{-i\pi s} \frac{\Gamma(\alpha_2 + s) \Gamma(\alpha_3 + s) \Gamma(\gamma_1 - s) \Gamma(\gamma_2 + v - s)}{\Gamma(\alpha_2 + \alpha_3 + \gamma_3 + m + s) \Gamma(\alpha_1 + \gamma_1 + \gamma_2 + v - s)} \\ = e^{-i\pi\alpha_2} \frac{\Gamma(\alpha_3 - \alpha_2) \Gamma(\gamma_1 + \alpha_2) \Gamma(\gamma_2 + v + \alpha_2)}{\Gamma(\alpha_3 + \gamma_3 + m) \Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + v)} \\ \cdot {}_3F_2 \left(\begin{matrix} \gamma_1 + \alpha_2, \gamma_2 + v + \alpha_2, 1 - \alpha_3 - \gamma_3 - m \\ 1 + \alpha_2 - \alpha_3, \alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + v \end{matrix}; 1 \right) \\ + e^{-i\pi\alpha_3} \frac{\Gamma(\alpha_2 - \alpha_3) \Gamma(\gamma_1 + \alpha_3) \Gamma(\gamma_2 + v + \alpha_3)}{\Gamma(\alpha_2 + \gamma_3 + m) \Gamma(\alpha_1 + \alpha_3 + \gamma_1 + \gamma_2 + v)} \\ \cdot {}_3F_2 \left(\begin{matrix} \gamma_1 + \alpha_3, \gamma_2 + v + \alpha_3, 1 - \alpha_2 - \gamma_3 - m \\ 1 - \alpha_2 + \alpha_3, \alpha_1 + \alpha_3 + \gamma_1 + \gamma_2 + v \end{matrix}; 1 \right)$$

Evaluating the v integral in (6.16) with the residue theorem finally yields

$$\begin{aligned}
 (6.17) \quad k_m = & \frac{\Gamma(\alpha_1 + \gamma_2)\Gamma(\alpha_2 + \gamma_3 + m)\Gamma(\alpha_3 + \gamma_3 + m)\Gamma(\alpha_4 + \gamma_1)}{\Gamma(1 - \alpha_4 - \gamma_4)\Gamma(m + 1)} \\
 & \cdot \sum_{\ell=0}^{\infty} \frac{\Gamma(\alpha_1 + \gamma_1 + \ell)\Gamma(1 - \alpha_4 - \gamma_4 + \ell)}{\Gamma(1 - \gamma_4 + \gamma_1 + \ell)\Gamma(\ell + 1)} \\
 & \cdot \left(e^{-i\pi\alpha_2} \frac{\Gamma(\alpha_3 - \alpha_2)\Gamma(\gamma_1 + \alpha_2)\Gamma(\gamma_2 + \ell + \alpha_2)}{\Gamma(\alpha_3 + \gamma_3 + m)\Gamma(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + \ell)} \right. \\
 & \cdot {}_3F_2 \left(\begin{matrix} \gamma_1 + \alpha_2, \gamma_2 + \ell + \alpha_2, 1 - \alpha_3 - \gamma_3 - m \\ 1 + \alpha_2 - \alpha_3, \alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + \ell \end{matrix}; 1 \right) \\
 & + e^{-i\pi\alpha_3} \frac{\Gamma(\alpha_2 - \alpha_3)\Gamma(\gamma_1 + \alpha_3)\Gamma(\gamma_2 + \ell + \alpha_3)}{\Gamma(\alpha_2 + \gamma_3 + m)\Gamma(\alpha_1 + \alpha_3 + \gamma_1 + \gamma_2 + \ell)} \\
 & \cdot {}_3F_2 \left(\begin{matrix} \gamma_1 + \alpha_3, \gamma_2 + \ell + \alpha_3, 1 - \alpha_2 - \gamma_3 - m \\ 1 - \alpha_2 + \alpha_3, \alpha_1 + \alpha_3 + \gamma_1 + \gamma_2 + \ell \end{matrix}; 1 \right) \Bigg)
 \end{aligned}$$

7. APPLICATIONS

7.1. The Frobenius method. The linear differential equation (2.1) can equivalently be written as a first order matrix differential equation

$$(7.1) \quad \theta Y(z) = A(z)Y(z)$$

where

$$A(z) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}, \quad Y(z) = \begin{pmatrix} y(z) \\ \theta y(z) \\ \vdots \\ \theta^{n-2} y(z) \\ \theta^{n-1} y(z) \end{pmatrix}$$

where

$$a_i = \frac{e_{n+1-i}(\gamma_1, \dots, \gamma_{n-1}) - z e_{n+1-i}(\alpha_1, \dots, \alpha_n)}{1 - z},$$

$e_i(x_1, \dots, x_k)$ is the elementary symmetric polynomial of degree i , and $y(z)$ is a solution to (2.1). For linearly independent solutions y_1, \dots, y_n to (2.1) we have linearly independent solution vectors Y_1, \dots, Y_n to (7.1). We collect these column vectors into a matrix $\Phi = (Y_1 \dots Y_n)$, known as a fundamental matrix, since it satisfies $\theta \Phi = A\Phi$. Near any regular singularity $z = z_0$ of (7.1), there exist a constant $n \times n$ matrix R , a real number $r > 0$, and an $n \times n$ matrix S of singly-valued holomorphic functions in the annulus $0 < |z - z_0| < r$ such that the fundamental matrix takes the following form [4]

$$\Phi(z) = S(z)(z - z_0)^R.$$

The matrix R is determined by A , e.g. if the eigenvalues of $A(0)$ do not differ by positive integers, then $R = A(0)$. The matrix $S(z)$ can be determined as follows. For the first row of $S(z)$ we make a power series ansatz and substitute it into (7.1). This yields recursion relations for the coefficients of the power series. These recursion relations can be solved after choosing a number of constants. This is known as the Frobenius method. The remaining rows of $S(z)$ are obtained from the first row by successively acting with θ on the first row.

Note that the fundamental matrix is not unique. Multiplication by any invertible constant $n \times n$ matrix C yields another fundamental matrix. We will use this freedom in the examples to choose bases which are easy to relate with the basis elements $y_j^*(z)$, $y_{ij}(z)$, $G_p(z)$, or $\xi_n(z)$ given in terms of integral representations.

7.2. The mirror quartic. The variation of polarized Hodge structure of the family $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$ of mirror quartics given as

$$\mathcal{X}_z = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4z^{-\frac{1}{4}}x_0x_1x_2x_3 = 0\} \subset \mathbb{P}^3$$

leads to a Picard–Fuchs equation which is a hypergeometric differential equation of order 3 with exponents [13]

$$\alpha_i = \frac{i}{4}, \quad \gamma_i = 0, \quad i = 1, 2, 3, \quad \beta_3 = \frac{1}{2}.$$

A fundamental system at $z = 0$ is $\Phi_0(z) = S_0(z)z^{R_0}4^{-4R_0}C_0$ with

$$R_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2\pi i} & 0 \\ 0 & 0 & \frac{1}{(2\pi i)^2} \end{pmatrix}$$

and

$$\begin{aligned} S_{0,11} &= 1 + \frac{3}{32}z + \frac{315}{8192}z^2 + \frac{5775}{262144}z^3 + O(z^4) \\ S_{0,12} &= \frac{13}{32}z + \frac{3069}{16384}z^2 + \frac{176005}{1572864}z^3 + O(z^4) \\ S_{0,13} &= \frac{169}{2048}z^2 + \frac{35841}{524288}z^3 + O(z^4) \end{aligned}$$

A fundamental system at $y = 1 - z$ is $\Phi_1(y) = S_1(y)y^{R_1}$ with

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

and

$$\begin{aligned} S_{1,11} &= 1 - \frac{1}{32}y^2 - \frac{131}{3840}y^3 + O(y^4) \\ S_{1,12} &= 1 + \frac{35}{48}y + \frac{665}{1152}y^2 + O(y^3) \\ S_{1,13} &= 1 + \frac{11}{24}y + \frac{39}{128}y^2 + \frac{1181}{5120}y^3 + O(y^4) \end{aligned}$$

Theorem 7.1. *The analytic continuation of $\Phi_0(z)$ to $z = 1$ is determined by $\Phi_0(z) = \Phi_1(1 - z)M_{10}$ with*

$$M_{10} = \begin{pmatrix} \frac{A}{2\sqrt{2\pi}} & -\frac{A}{4\pi i} & 0 \\ \frac{2}{\sqrt{2\pi}}\left(\frac{3A}{64} + \frac{1}{A}\right) & -\frac{1}{\pi i}\left(\frac{3A}{64} - \frac{1}{A}\right) & 0 \\ -\frac{2}{\sqrt{2\pi}} & 0 & -\frac{1}{\sqrt{2\pi}} \end{pmatrix}$$

where $A = \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}$.

Proof. By evaluating the residues in the definition of $G_p(z)$ closing the contour to the right one finds that

$$\begin{aligned} G_1(z) &= \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)\Phi_{0,11}, \\ \frac{1}{2\pi i}G_2(z) &= -\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)\Phi_{0,12}, \\ \frac{1}{(2\pi i)^2}G_3(z) &= \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)\left(\Phi_{0,13} + \frac{1}{2}\Phi_{0,12} - \frac{1}{2}\Phi_{0,11}\right). \end{aligned}$$

Corollary 6.1 now yields

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)\Phi_{0,11} = g_0(0)\Phi_{1,11} + g_1(0)\Phi_{1,12} + g_0\left(\frac{1}{2}\right)\Phi_{1,13}$$

where the expansion coefficients are determined by (6.3), (6.4) and (6.5) as

$$\begin{aligned} g_0(0) &= \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})^2}{\Gamma(\frac{3}{4})} {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ \frac{3}{4}, 1 \end{matrix}; 1\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{2\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} \\ g_1(0) &= -\frac{\Gamma(\frac{5}{4})\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{4})} {}_3F_2\left(\begin{matrix} -\frac{1}{2}, \frac{1}{4}, \frac{1}{4} \\ \frac{3}{4}, 1 \end{matrix}; 1\right) \\ &= 2\Gamma(\frac{1}{2}) \left(\frac{3\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{64\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} + \frac{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})} \right) \\ g_0(\frac{1}{2}) &= -2\Gamma(\frac{1}{2}) \end{aligned}$$

from which the first row of M_{01} follows using $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \sqrt{2}\pi$.

Theorem 6.5 (see Example 6.7) yields

$$-2\pi i \Gamma(\frac{1}{4})\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})\Phi_{0,12} = h_0\Phi_{1,11} + h_1\Phi_{1,12}$$

where the expansion coefficients are determined by (6.12) as

$$\begin{aligned} h_0 &= \Gamma(\frac{1}{4})^2\Gamma(\frac{1}{2})^2 {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ \frac{3}{4}, 1 \end{matrix}; 1\right) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{2\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} \\ h_1 &= \frac{1}{6}\Gamma(\frac{1}{4})^2\Gamma(\frac{1}{2})^2 {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ \frac{7}{4}, 1 \end{matrix}; 1\right) \\ &= 2\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})\Gamma(\frac{3}{4}) \left(\frac{3\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{64\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} - \frac{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})} \right) \end{aligned}$$

from which the second row of M_{01} follows.

It remains to explain the evaluation of the various ${}_3F_2$ at 1. The basic formula is Dixon's identity [16]:

$$\begin{aligned} {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ 1+a_1-a_2, 1+a_1-a_3 \end{matrix}; 1\right) \\ = \frac{\Gamma(1+\frac{a_1}{2})\Gamma(1+\frac{a_1}{2}-a_2-a_3)\Gamma(1+a_1-a_2)\Gamma(1+a_1-a_3)}{\Gamma(1+a_1)\Gamma(1+a_1-a_2-a_3)\Gamma(1+\frac{a_1}{2}-a_2)\Gamma(1+\frac{a_1}{2}-a_3)} \end{aligned}$$

This identity can be used to evaluate $g_0(0)$ and h_0 . There is a generalization of this identity due to Lavoie et al. [12]. The two cases we need for $g_1(0)$ and h_1 are

$$\begin{aligned} {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ a_1-a_2, 1+a_1-a_3 \end{matrix}; 1\right) &= \frac{2^{-2a_3}\Gamma(a_1-a_2)\Gamma(a_1-a_3+1)}{\Gamma(a_1-2a_3+1)\Gamma(a_1-a_2-a_3+1)} \\ &\cdot \left(\frac{\Gamma(\frac{a_1}{2}-a_3+\frac{1}{2})\Gamma(\frac{a_1}{2}-a_2-a_3+1)}{\Gamma(\frac{a_1}{2}+\frac{1}{2})\Gamma(\frac{a_1}{2}-a_2)} + \frac{\Gamma(\frac{a_1}{2}-a_3+1)\Gamma(\frac{a_1}{2}-a_2-a_3+\frac{1}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{a_1}{2}-a_2+\frac{1}{2})} \right), \\ {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ 2+a_1-a_2, 1+a_1-a_3 \end{matrix}; 1\right) &= \frac{2^{1-2a_2}\Gamma(a_1-a_3+1)\Gamma(a_1-a_2+2)\Gamma(a_2-1)}{\Gamma(a_1-2a_2+2)\Gamma(a_1-a_2-a_3+2)\Gamma(a_2)} \\ &\cdot \left(-\frac{\Gamma(\frac{a_1}{2}-a_2+\frac{3}{2})\Gamma(\frac{a_1}{2}-a_3-a_2+2)}{\Gamma(\frac{a_1}{2}+\frac{1}{2})\Gamma(\frac{a_1}{2}-a_3+1)} + \frac{\Gamma(\frac{a_1}{2}-a_2+1)\Gamma(\frac{a_1}{2}-a_3-a_2+\frac{3}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{a_1}{2}-a_3+\frac{1}{2})} \right), \end{aligned}$$

respectively.

Finally, we have $\Phi_{1,13}(y) = \xi_3(z)$. Proposition 5.1 with $\psi(x) = i \prod_{\nu=1}^3 (x - \omega^{-\nu})$, $\omega^4 = 1$ yields $\psi(1) = 4i$, $\psi'(1) = 6i$, $\frac{\psi''(1)}{2!} = 4i$ so that

$$\Phi_{1,13}(y) = -\frac{\Gamma(\frac{3}{2})}{\pi} (2y_1^*(z) - 3y_2^*(z) + 2y_3^*(z)).$$

Lemma 2.3(2) then yields $\Phi_{0,13}(z) = -\frac{1}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}\Phi_{1,13}(y)$. \square

7.3. The mirror quintic. The variation of polarized Hodge structure of the family $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$ of mirror quintics given as

$$\mathcal{X}_z = \{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5z^{-\frac{1}{5}}x_0x_1x_2x_3x_4 = 0\} \subset \mathbb{P}^4$$

leads to a Picard–Fuchs equation which is a hypergeometric differential equation of order 4 with exponents [3]

$$\alpha_i = \frac{i}{5}, \quad \gamma_i = 0, \quad i = 1, 2, 3, 4, \quad \beta_4 = 1.$$

A fundamental system at $z = 0$ is $\Phi_0(z) = S_0(z)z^{R_0}5^{-5R_0}C_0$ with

$$R_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2\pi i} & 0 & 0 \\ 0 & 0 & \frac{1}{(2\pi i)^2} & 0 \\ 0 & 0 & 0 & \frac{1}{(2\pi i)^3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -\frac{25}{12} & \frac{200}{(2\pi i)^3}\zeta(3) \\ 0 & 1 & \frac{5}{2} & -\frac{25}{12} \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

and

$$\begin{aligned} S_{0,11} &= 1 + \frac{24}{625}z + \frac{4536}{390625}z^2 + O(z^3) \\ S_{0,12} &= \frac{154}{625}z + \frac{32409}{390625}z^2 + O(z^3) \\ S_{0,13} &= \frac{23}{125}z + \frac{168327}{1562500}z^2 + O(z^3) \\ S_{0,14} &= -\frac{46}{125}z - \frac{26387}{312500}z^2 + O(z^3) \end{aligned}$$

The choice of C_0 follows from [9]. A fundamental system at $y = 1 - z$ is $\Phi_1(y) = S_1(y)y^{R_1}C_1$ with

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad C_1 = \frac{\sqrt{5}}{4\pi^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} S_{1,11} &= 1 + \frac{2}{625}y^3 + \frac{97}{18750}y^4 + \frac{2971}{468750}y^5 + O(y^6) \\ S_{1,12} &= 1 + \frac{7}{10}y + \frac{41}{75}y^2 + \frac{1133}{2500}y^3 + \frac{6089}{15625}y^4 + \frac{160979}{468750}y^5 + O(y^6) \\ S_{1,13} &= -\frac{23}{360}y^2 - \frac{6397}{60000}y^3 - \frac{333323}{2500000}y^4 - \frac{33777511}{225000000}y^5 + O(y^6) \\ S_{1,14} &= 1 + \frac{37}{30}y + \frac{2309}{1800}y^2 + \frac{286471}{225000}y^3 + \frac{41932661}{33750000}y^4 + \frac{237108737}{196875000}y^5 + O(y^6) \end{aligned}$$

Theorem 7.2. *Let $\Phi_{z_0}(z)$ be the fundamental matrices near $z_0 = 0, 1$. The analytic continuation of $\Phi_0(z)$ to $z = 1$ is determined by $\Phi_0(z) = \Phi_1(1 - z)M_{10}$ with*

$$M_{10} = \begin{pmatrix} l_0 & -\frac{h_0}{2\pi i} & \frac{5k_0}{(2\pi i)^2} & 0 \\ w_0 & -\frac{h_1}{2\pi i} & \frac{5k_1}{(2\pi i)^2} & 2\pi i \\ 1 & 0 & 0 & 0 \\ w_1 - \frac{7}{10}w_0 & -\frac{h_2}{2\pi i} + \frac{7}{10}\frac{h_1}{2\pi i} & \frac{5k_2}{(2\pi i)^2} - \frac{7}{10}\frac{5k_1}{(2\pi i)^2} & 0 \end{pmatrix}$$

where the real constants $l_0, w_0, w_1, h_0, h_1, h_2$ and the complex constants k_0, k_1, k_2 are explicitly given in the proof.

Proof. By evaluating the residues in the definition of $G_p(z)$ closing the contour to the right one finds that

$$\begin{aligned} G_1(z) &= \frac{4\pi^2}{\sqrt{5}}\Phi_{0,11}, \\ \frac{1}{2\pi i}G_2(z) &= -\frac{4\pi^2}{\sqrt{5}}\Phi_{0,12}, \\ \frac{1}{(2\pi i)^2}G_3(z) &= \frac{4\pi^2}{\sqrt{5}}\Phi_{0,13}, \\ \frac{1}{(2\pi i)^3}G_4(z) &= \frac{4\pi^2}{\sqrt{5}}\left(\frac{1}{5}\Phi_{0,14} - \frac{1}{5}\Phi_{0,13} + \Phi_{0,12}\right). \end{aligned}$$

where the prefactor comes from $\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})\Gamma(\frac{3}{5})\Gamma(\frac{4}{5}) = \frac{4\pi^2}{\sqrt{5}}$.

Corollary 6.2 yields

$$\frac{4\pi^2}{\sqrt{5}}\Phi_{0,11} = l_0 + w_0(1-z) + q_0(1-z)\log(1-z) + w_1(1-z)^2 + O((1-z)^3)$$

and since $\Phi_{1,12} = (1-z) + \frac{7}{10}(1-z)^2 + O((1-z)^3)$ this expression can be written as

$$\Phi_{0,11} = l_0\Phi_{1,11} + w_0\Phi_{1,12} + q_0\Phi_{1,13} + (w_1 - \frac{7}{10}w_0)\Phi_{1,14}$$

from which the first row of M_{01} follows. The explicit values of the coefficients are obtained as follows. From (6.2) we get

$$A^{(4)}(k) = \frac{(\frac{3}{5})_k(\frac{2}{5})_k}{\Gamma(1+k)} {}_3F_2\left(\frac{1}{5}, \frac{1}{5}, -k; \frac{3}{5}, \frac{3}{5} - k; 1\right)$$

Then, (6.6), (6.7), (6.8) and (6.9) yield

$$(7.2) \quad \begin{aligned} l_0 &= \frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{3}{5})} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{5}+k)\Gamma(\frac{2}{5}+k)}{\Gamma(\frac{6}{5}+k)\Gamma(\frac{7}{5}+k)} {}_3F_2\left(\frac{1}{5}, \frac{1}{5}, -k; \frac{3}{5}, \frac{3}{5} - k; 1\right) \\ q_0 &= 1, \\ w_0 &= -\psi(1) - \psi(2) + \psi(\frac{6}{5}) + \psi(\frac{7}{5}) - \sum_{k=1}^{\infty} \frac{(\frac{3}{5})_k(\frac{2}{5})_k}{k(\frac{6}{5})_k(\frac{7}{5})_k} {}_3F_2\left(\frac{1}{5}, \frac{1}{5}, -k; \frac{3}{5}, \frac{3}{5} - k; 1\right) \\ w_1 &= -\frac{21}{25} \left(\psi(2) + \psi(3) - \psi(\frac{11}{5}) - \psi(\frac{12}{5}) \right. \\ &\quad \left. - \frac{1}{6}(\psi(1) + \psi(3) - \psi(\frac{11}{5}) - \psi(\frac{12}{5})) \right. \\ &\quad \left. - \sum_{k=2}^{\infty} \frac{(\frac{3}{5})_k(\frac{2}{5})_k}{k(k-1)(\frac{6}{5})_k(\frac{7}{5})_k} {}_3F_2\left(\frac{1}{5}, \frac{1}{5}, -k; \frac{3}{5}, \frac{3}{5} - k; 1\right) \right) \end{aligned}$$

Next, Theorem 6.5 (see Example 6.8) yields

$$G_2(z) = h_0 + h_1(1-z) + h_2(1-z)^2 + O((1-z)^3)$$

Again, since $\Phi_{1,12} = (1-z) + \frac{7}{10}(1-z)^2 + O((1-z)^3)$, we have that

$$\Phi_{0,12} = -\frac{h_0}{2\pi i}\Phi_{1,11} - \frac{h_1}{2\pi i}\Phi_{1,12} - \left(\frac{h_2}{2\pi i} - \frac{7}{10}\frac{h_1}{2\pi i}\right)\Phi_{1,14}$$

from which the second row of M_{01} follows. The explicit values of the coefficients h_m are determined by (6.14) as

$$(7.3) \quad \begin{aligned} h_0 &= \Gamma(\frac{1}{5})^2\Gamma(\frac{2}{5})^2\Gamma(\frac{4}{5}) \sum_{\ell=0}^{\infty} \frac{(\frac{1}{5})_{\ell}(\frac{2}{5})_{\ell}}{(\frac{3}{5})_{\ell}\ell!} {}_3F_2\left(-\ell, \frac{3}{5}, \frac{4}{5}; 1, 1; 1\right) \\ h_1 &= \frac{2}{15}\Gamma(\frac{1}{5})^2\Gamma(\frac{2}{5})^2\Gamma(\frac{4}{5}) \sum_{\ell=0}^{\infty} \frac{(\frac{1}{5})_{\ell}(\frac{2}{5})_{\ell}}{(\frac{8}{5})_{\ell}\ell!} {}_3F_2\left(-\ell, \frac{3}{5}, \frac{4}{5}; 1, 1; 1\right) \\ h_2 &= \frac{7}{100}\Gamma(\frac{1}{5})^2\Gamma(\frac{2}{5})^2\Gamma(\frac{4}{5}) \sum_{\ell=0}^{\infty} \frac{(\frac{1}{5})_{\ell}(\frac{2}{5})_{\ell}}{(\frac{13}{5})_{\ell}\ell!} {}_3F_2\left(-\ell, \frac{3}{5}, \frac{4}{5}; 1, 1; 1\right) \end{aligned}$$

Moreover, Theorem 6.5 (see Example 6.10) also yields

$$G_3(z) = k_0 + k_1(1-z) + k_2(1-z)^2 + O((1-z)^3)$$

Hence, by the same reasoning as above

$$\Phi_{0,13} = \frac{5k_0}{(2\pi i)^2}\Phi_{1,11} + \frac{5k_1}{(2\pi i)^2}\Phi_{1,12} + \left(\frac{5k_2}{(2\pi i)^2} - \frac{7}{10}\frac{5k_1}{(2\pi i)^2}\right)\Phi_{1,14}$$

from which the third row of M_{01} follows. The explicit values of the coefficients k_m are determined by (6.17) as

$$\begin{aligned}
 (7.4) \quad k_0 &= \Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{4}{5}\right) \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{1}{5} + \ell\right)^2}{\Gamma(\ell+1)^2} \\
 &\quad \cdot \left(e^{-i\pi\frac{2}{5}} \frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{2}{5} + \ell\right)}{\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{3}{5} + \ell\right)} {}_3F_2\left(\frac{2}{5}, \frac{2}{5} + \ell, \frac{2}{5}; \frac{4}{5}, \frac{3}{5} + \ell; 1\right) \right. \\
 &\quad \left. + e^{-i\pi\frac{3}{5}} \frac{\Gamma\left(-\frac{1}{5}\right)\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{3}{5} + \ell\right)}{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{4}{5} + \ell\right)} {}_3F_2\left(\frac{3}{5}, \frac{3}{5} + \ell, \frac{3}{5}; \frac{6}{5}, \frac{4}{5} + \ell; 1\right) \right) \\
 k_1 &= \Gamma\left(\frac{4}{5}\right)\Gamma\left(\frac{7}{5}\right)\Gamma\left(\frac{8}{5}\right) \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{1}{5} + \ell\right)^2}{\Gamma(\ell+1)^2} \\
 &\quad \cdot \left(e^{-i\pi\frac{2}{5}} \frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{2}{5} + \ell\right)}{\Gamma\left(\frac{8}{5}\right)\Gamma\left(\frac{3}{5} + \ell\right)} {}_3F_2\left(\frac{2}{5}, \frac{2}{5} + \ell, -\frac{3}{5}; \frac{4}{5}, \frac{3}{5} + \ell; 1\right) \right. \\
 &\quad \left. + e^{-i\pi\frac{3}{5}} \frac{\Gamma\left(-\frac{1}{5}\right)\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{3}{5} + \ell\right)}{\Gamma\left(\frac{7}{5}\right)\Gamma\left(\frac{4}{5} + \ell\right)} {}_3F_2\left(\frac{3}{5}, \frac{3}{5} + \ell, -\frac{2}{5}; \frac{6}{5}, \frac{4}{5} + \ell; 1\right) \right) \\
 k_2 &= \frac{1}{2}\Gamma\left(\frac{4}{5}\right)\Gamma\left(\frac{12}{5}\right)\Gamma\left(\frac{13}{5}\right) \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{1}{5} + \ell\right)^2}{\Gamma(\ell+1)^2} \\
 &\quad \cdot \left(e^{-i\pi\frac{2}{5}} \frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{2}{5} + \ell\right)}{\Gamma\left(\frac{13}{5}\right)\Gamma\left(\frac{3}{5} + \ell\right)} {}_3F_2\left(\frac{2}{5}, \frac{2}{5} + \ell, -\frac{8}{5}; \frac{4}{5}, \frac{3}{5} + \ell; 1\right) \right. \\
 &\quad \left. + e^{-i\pi\frac{3}{5}} \frac{\Gamma\left(-\frac{1}{5}\right)\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{3}{5} + \ell\right)}{\Gamma\left(\frac{12}{5}\right)\Gamma\left(\frac{4}{5} + \ell\right)} {}_3F_2\left(\frac{3}{5}, \frac{3}{5} + \ell, -\frac{7}{5}; \frac{6}{5}, \frac{4}{5} + \ell; 1\right) \right)
 \end{aligned}$$

Finally, we have $\Phi_{1,12}(y) = \frac{\sqrt{5}}{4\pi^2}\xi_4(z)$. Proposition 5.1 with $\psi(x) = \prod_{\nu=1}^4 (x - \omega^{-\nu})$, $\omega^5 = 1$ yields $\psi(1) = 5$, $\psi'(1) = 10$, $\frac{\psi''(1)}{2!} = 10$, $\frac{\psi'''(1)}{3!} = 5$, so that

$$\Phi_{1,12}(y) = \frac{5}{2\pi i} (y_4^*(z) - 2y_3^*(z) + 2y_2^*(z) - y_1^*(z)).$$

Lemma 2.3(2) then yields $\Phi_{0,14}(z) = 2\pi i \Phi_{1,12}(y)$. This result has also been obtained through a monodromy argument in [3]. \square

At present, we are not aware of any identities that help evaluating the infinite sums in l_0 , w_m , h_m and k_m . Numerical evaluation shows, however, that the following identity should hold:

$$\Im k_m = \pi i h_m, m = 0, 1, 2.$$

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